18.103 MIDTERM I

Wednesday, March 20, 2007

Name: ——————————————————————

Numeric Student ID: ———————————————

Instructor's Name: ———————————————

I agree to abide by the terms of the honor code:

Signature: ————————————————————

Instructions: Print your name, student ID number and instructor's name in the space provided. During the test you may not use notes, books or calculators. Read each question carefully and **show all your work**; full credit cannot be obtained without sufficient justification for your answer unless explicitly stated otherwise. Underline your final answer to each question. There are 5 questions. You have 55 minutes to do all the problems.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5 (Bonus)		5
Total		40

1. State both Borel-Cantelli Lemmas, being careful to define the terms used, and give the proof of one of them (your choice).

   Solution:

   You can look this up. See section 1.4 of Adams-Guillemin. Most people did well on the two proofs questions. If problems occurred on this question, it was typically a confusion about the order of intersection and union on the definition of "infinitely often" or else a confusion about what it means for a countable collection of functions to be independent (every finite subcollection is independent).

2. Prove the Lebesgue Dominated Convergence Theorem, assuming Fatou's Lemma. Recall the LDCT states:

   Let $\{f_i\}$ be a sequence of measurable functions such that, for a given measurable set E,
   $$\lim_{n\to\infty} f_n(x) \text{ exists for all } x \in E,$$
   and suppose that g is integrable on E with $g \geq |f_n|$ for all n. Then show that $f(x) = \lim_{n\to\infty} f_n(x)$ is integrable and
   $$\int_E f \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu$$

Solution:

You can look this up, too. It's page 78 of Adams-Guillemin. Again, people did very well on this, with the very few mistakes happening from a manipulation of the integral involving $g - f_n$ and how lim inf and lim sup interact with minus signs.

3. Let f be a non-negative function on a measurable set E. Prove that $\int_E f\ d\mu = 0$ if and only if $f = 0$ almost everywhere on E.

   Solution:

   Suppose $f = 0$ almost everywhere on E. Then any simple function $\psi \leq f$ must also be 0 almost everywhere. Direct computation of the Lebesgue integral for this simple function ψ gives

   $$I_E(\psi) = \sum_i c_i \mu(E_i) = 0$$

   since, for every i, either $c_i = 0$ or else $\mu(E_i) = 0$. Then by definition of the integral of a non-negative function as the sup over all $\psi \leq f$, $\int_E f d\mu = 0$.

   In the reverse direction, if the integral is 0, then we show the sets of the form

   $$E_\epsilon = \{x \in E \mid f(x) > \epsilon\},$$

   for some sequence of $\{\epsilon\}$ converging to 0, can't have positive measure. For example, consider $E_{1/n}$. By Chebyshev's inequality,

   $$\mu(E_n) \leq n \int_E f d\mu = 0 \quad \text{for all } n \in \mathbb{N}$$

   so that the set we care about,

   $$F = \{x \in E \mid f(x) \neq 0\} = \bigcup_{n=1}^{\infty} E_{1/n},$$

   and then by countable subadditivity, we have

   $$\mu(F) \leq \sum_{n=1}^{\infty} \mu(E_{1/n}) = 0.$$

4. If $\{f_n\}$ is a sequence of measurable functions with each $f_n : X \to \mathbb{R}$, then show that

$$\left\{ x \mid \lim_{n \to \infty} f_n(x) \text{ exists} \right\}$$

is measurable.

Solution:

There are basically two proofs. One is to define an auxilliary function

$$g(x) = \limsup f_n(x) - \liminf f_n(x).$$

Recalling all our properties of measurable functions, it is not hard to show that if the sequence of f_n are measurable, then so are the $\liminf f_n$ and $\limsup f_n$. Then noting that $g(x)$ is non-negative, and the set

$$\{x \mid g(x) \leq 0\} = \{x \mid g(x) = 0\}$$

is measurable, then we're done, since this is precisely the set x for which the limit of the f_n exist.

Alternately, noting that the set of x for which the limit exists can be expressed, via Cauchy sequences, as

$$\{x \mid \text{For any } \epsilon > 0, \text{ there exists an } N \text{ such that } |f_n(x) - f_m(x)| < \epsilon, \text{ for all } m, n \geq N\}$$

Now again choosing a sequence $\{\epsilon_k\}$ converging to 0 as $k \to \infty$, We may express this set as a collection of countable unions with respect to m, n, N and intersections with respect to k (in the appropriate order. The intersection over k should be on the outside). Since the basic set (for any fixed m, n, ϵ)

$$\{x \mid |f_n(x) - f_m(x)| < \epsilon\}$$

is measurable because each of the f_n's is measurable, and moreover, countable unions and intersections of measurable sets are measurable, then the set in question is indeed measurable.

5. (Bonus) Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \, \cos(nx) \, d\mu_L(x) = 0.$$

Solution:

I'll just give a hint on this one and then write the rest of the solution up tomorrow for those that want to work on this a bit more. First, no convergence theorem will help you here, as the limit of the functions inside the integral doesn't exist. Second, one always wants to reduce integrals to questions about simple functions, but in this case, that's not quite enough. Try to prove it first for step functions.