18.103 Problem Set 8 Partial Solutions

Sawyer Tabony

3.3.8 This problem is actually solved by combining the results of the previous two problems. In 3.3.6 you showed that the S_n's converge to a function H in the $\mathcal{L}^2$ norm. In 3.3.7 you proved a bound on the size of the measure of

$$A(\varepsilon, n, m) = \{\omega \in I : |S_k(\omega) - S_n(\omega)| > \varepsilon \text{ for some } k \text{ between } m \text{ and } n\}$$

for fixed $m > n$, $\varepsilon > 0$.

Now we are asked to show that the S_n's converge pointwise almost everywhere to H. Let's prove this by contradiction. So we assume not, that there is some set of positive measure B, with

$$\forall \omega \in B,\ S_n(\omega) \text{ does not converge pointwise},$$

or,

$$\forall \omega \in B,\ \exists \varepsilon(\omega) > 0 \text{ such that } \forall N \in \mathbb{N}, \exists n > N \text{ with } |S_n(\omega) - H(\omega)| > \varepsilon.$$

Now $\varepsilon(\omega)$ is a positive function on B, a set of positive measure, so using the same old trick (B is the union of $\varepsilon^{-1}((\frac{1}{n}, \infty))$), there is some fixed ε that works for a subset C of B of positive measure. Let $\mu(C) = \delta > 0$. We note that for any $\omega \in C$, for that ε and any $n \in \mathbb{N}$, $\exists m \in \mathbb{N}$ such that $\omega \in A(\varepsilon, n, m)$.

Now applying problem 3.3.6, since S_n converges to H in the $\mathcal{L}^2$ norm, it is Cauchy, so we can choose an $N \in \mathbb{N}$ such that for any $m > n > N$,

$$\int (S_m - S_n)^2 d\mu < \frac{1}{2}\varepsilon^2\delta.$$

Thus, for any $m > n > N$, by problem 3.3.7,

$$\mu(A(\varepsilon, n, m)) \leq \frac{1}{\varepsilon^2}\int (S_m - S_n)^2 d\mu < \frac{1}{\varepsilon^2} \cdot \frac{1}{2}\varepsilon^2\delta = \frac{1}{2}\delta.$$

Now, we send $m \longrightarrow \infty$. Since the sets are nested, we have

$$\lim_{m \longrightarrow \infty} \mu(A(\varepsilon, n, m)) = \mu\left(\bigcup_{m=n+1}^{\infty} A(\varepsilon, n, m)\right) \leq \frac{1}{2}\delta.$$

But we had above that

$$C \subseteq \bigcup_{m=n+1}^{\infty} A(\varepsilon, n, m),$$

so

$$\delta = \mu(C) \leq \lim_{m \longrightarrow \infty} \mu(A(\varepsilon, n, m)) \leq \frac{1}{2}\delta.$$

This is our contradiction, showing that $S_n(\omega)$ converges for almost all ω.