18.103 Problem Set 6 Partial Solutions

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2.5.3 Many people had confusion with how to define $\mathcal{M} \times \mathcal{N}$. Instead of just being the Cartesian product of the two sets, it is the smallest σ-field containing all product sets $A \times B$ for $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

a) We want to show

$$(\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3 = \mathcal{M}_1 \times (\mathcal{M}_2 \times \mathcal{M}_3),$$

and we can do this by showing each is equal to the smallest σ-field containing the product sets $A_1 \times A_2 \times A_3$ for $A_i \in \mathcal{M}_i$. First, we show the left side is equal to this σ-field; the right side will follow in exactly the same manner.

Given $A_i \in \mathcal{M}_i$, $A_1 \times A_2 \in \mathcal{M}_1 \times \mathcal{M}_2$ by the definition of $\mathcal{M}_1 \times \mathcal{M}_2$. Thus,

$$A_1 \times A_2 \times A_3 \in (\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3.$$

So the left side contains the triple-product sets. Now consider an element of $(\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3$ of the form $B \times A_3$ for $B \in \mathcal{M}_1 \times \mathcal{M}_2$. So B is in every σ-field containing all the product sets $A_1 \times A_2$, for $A_i \in \mathcal{M}_i$. Taking the Cartesian product with A_3, we see that every σ-field containing all the product sets $A_1 \times A_2 \times A_3$ contains $B \times A_3$. Therefore the left side above is the smallest σ-field containing the triple-product sets. A similar argument works for the other side, so the two sides must be equal.

2.5.7 We are given a function $h : X \times Y \longrightarrow \mathbb{R}$ and we have to show it is measurable. Notice this isn't the statement that the product of measurable functions is measurable, because the two functions we are multiplying have different domains, so instead of $h(x) = f(x)g(x)$, for $x \in X$, we have $h(x, y) = f(x)g(y)$, for $x \in X$ and $y \in Y$.

a) So we want h to be measurable, that is, for every $a \in \mathbb{R}$, we want $h^{-1}((a, \infty))$ to be measurable. For $a \geq 0$, we have:

$$\begin{aligned} h^{-1}((a,\infty)) &= \{(x,y) \in X \times Y | f(x)g(y) > a\} = \\ &= \bigcup_{r \in \mathbb{Q}^+} \{(x,y) \in X \times Y | f(x) > ar, g(y) > r^{-1}\} \cup \{(x,y) \in X \times Y | f(x) < -ar, g(y) < -r^{-1}\} \\ &= \bigcup_{r \in \mathbb{Q}^+} \left[f^{-1}((ar,\infty)) \times g^{-1}((r^{-1},\infty))\right] \cup \left[f^{-1}((-\infty,-ar)) \times g^{-1}((-\infty,-r^{-1}))\right]. \end{aligned}$$

By the measurability of f and g, we get that this is a countable union of products of measurable sets. Since these products are necessarily in $\mathcal{M} \times \mathcal{N}$, by the definition of $\times$, their countable union is as well, so this set is union is measurable. A similar union can be made for a negative, so we are done, and h is measurable on $X \times Y$.