18.103 Problem Set 5 Partial Solutions

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2.3.9 We are given an integrable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose integral over any interval is zero. We know we can write f as $f_+ - f_-$ for nonnegative integrable functions f_+ and f_-. Suppose

$$\int_{\mathbb{R}} f_+ d\mu_L > 0.$$

By definition of integrable, this integral is finite. Then, on some (finite) interval I, we have

$$0 < \int_I f_+ d\mu_L = D < \infty.$$

Now, we have $f = f_+ - f_-$, so by the hypothesis,

$$0 = \int_I f\ d\mu_L = \int_I f_+ d\mu_L - \int_I f_- d\mu_L = D - \int_I f_- d\mu_L \Longrightarrow \int_I f_- d\mu_L = D.$$

Now consider the sets $A_n = \{x \in I; f_- > n\}$. It is clear that

$$\int_I f_- d\mu_L = D \Longrightarrow \mu_L(A_n) < \frac{D}{n} \Longrightarrow \mu_L(A_n) \longrightarrow 0.$$

Therefore, because the A_n are nested,

$$\lim_{n \longrightarrow \infty} \int_{A_n} f_- d\mu_L = 0.$$

So for some N,

$$\int_{A_N} f_- d\mu_L < \frac{D}{2}.$$

Let $\varepsilon = \mu_L(A_N)$. Then for any measurable subset B of I with $\mu_L(B) \leq \varepsilon$, we have

$$\int_B f_- d\mu_L = \int_{B \cap A_N} f_- d\mu_L + \int_{B \setminus A_N} f_- d\mu_L \leq \int_{B \cap A_N} f_- d\mu_L + \int_{B \setminus A_N} N\ d\mu_L =$$
$$= \int_{B \cap A_N} f_- d\mu_L + \mu(B \setminus A_N) \cdot N \leq \int_{B \cap A_N} f_- d\mu_L + \mu(A_N \setminus B) \cdot N =$$
$$= \int_{B \cap A_N} f_- d\mu_L + \int_{A_N \setminus B} N\ d\mu_L \leq \int_{B \cap A_N} f_- d\mu_L + \int_{A_N \setminus B} f_- d\mu_L = \int_{A_N} f_- d\mu_L < \frac{D}{2}.$$

Similarly, we can choose ε to be small enough so that over any measurable subset B of I with $\mu_L(B) \leq \varepsilon$, the integral of f_+ over B is less than $\frac{D}{2}$.

Now we use a lemma proved in a previous to cleverly produce an interval on which f has a positive integral. For problem 2.2.7 we showed that any measurable subset of a finite interval can be approximated arbitrarily by a finite union of intervals. Let us approximate $T_+ = \{x \in I; f_+(x) > 0\}$ by a union of disjoint intervals $\{J_i\}_{i=1}^M$, so that

$$\mu_L \left(S \left(T_+, \bigcup_{i=1}^{M} J_i \right) \right) < \varepsilon.$$

Then we take the integral of f over all the J_i:

$$0 = \sum_{i=1}^{M} \int_{J_i} f \, d\mu_L = \int_{\cup J_i} f \, d\mu_L = \int_{T_+} f \, d\mu_L + \int_{\cup J_i \setminus T_+} f \, d\mu_L - \int_{T_+ \setminus (\cup J_i)} f \, d\mu_L = D - \int_{\cup J_i \setminus T_+} f_- d\mu_L - \int_{T_+ \setminus (\cup J_i)} f_+ d\mu_L.$$

But each of these final two integrals is less than $\frac{D}{2}$ in absolute value. Therefore D minus them cannot be zero. This is our contradiction, so our assumption must be false. Thus we have shown that

$$\int_{\mathbb{R}} f_+ d\mu_L = 0.$$

Since this is a nonnegative function, this shows that f_+ is zero almost everywhere. Similarly, f_- is zero almost everywhere (follows by applying the above argument to $-f$). So f is zero almost everywhere.