18.103 Problem Set 4 Partial Solutions

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2.2.7 First let us prove the following.

Lemma. Given some measurable $A \subseteq J$ for J a finite interval, $\exists \{I_j\}_{j=1}^{N}$ intervals such that for $B = \cup_{j=1}^{n} I_j$, $\mu(S(A, B)) < \varepsilon$.

We know that $\mu(A) \leq \mu(J) < \infty$, so $\mu(A)$ is finite. By measurability, $\mu(A) = \mu^*(A)$, so we can choose a countable set of intervals $\{I_j\}_{j=1}^{\infty}$ such that $A \subseteq \cup_{j=1}^{\infty} I_j$ and

$$\sum_{j=1}^{\infty} \mu(I_j) < \mu(A) + \frac{\varepsilon}{2}.$$

So in particular, this infinite sum converges. Therefore, $\exists N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty} \mu(I_j) < \frac{\varepsilon}{2}.$$

Thus we have, for $B = \cup_{j=1}^{N} I_j$,

$$\mu(S(A, B)) \leq \mu(A - B) + \mu(B - A) \leq \mu\left(\bigcup_{j=N+1}^{\infty} I_j\right) + \mu\left(\bigcup_{j=1}^{\infty} I_j\right) - \mu(A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

So we are given a simple function $s : J \longrightarrow \mathbb{R}$, for $J \subset \mathbb{R}$ a finite interval. If s takes on the values $c_1 < c_2 < \ldots < c_r$ on the measurable sets $A_1, A_2, \ldots, A_r$ that partition J, by the lemma we can choose finite unions of intervals B_i such that

$$\mu(S(A_i, B_i)) < \frac{\varepsilon}{r \cdot (|c_1| + |c_r|)}.$$

Then we may define a simple function f to be c_1 on B_1, c_2 on $B_2 \setminus B_1$, c_3 on $B_3 \setminus (B_1 \cup B_2)$, and so forth, letting $f = 0$ on $J \setminus (\cup B_i)$. Then f is a step function because the finite union and difference of intervals is a finite union of intervals. Then we can bound $|s - f| \leq |c_1| + |c_r|$ everywhere on J, and the set $T = \{x \in J | f(x) \neq s(x)\}$ is contained in

$$\bigcup_{i=1}^{r} S(A_i, B_i) \cup \bigcup_{1 \leq i < j \leq r} (B_i \cap B_j).$$

But any element $x \in B_i \cap B_j$ cannot be in both A_i and A_j, and so must be in either $B_i \setminus A_i$ or $B_j \setminus A_j$, so the second union is contained in the first. Therefore,

$$\int_J |s - f| d\mu_L = \int_T |s - f| d\mu_L \leq \int_T (|c_1| + |c_r|) d\mu_L = (|c_1| + |c_r|) \cdot \mu(T) \leq$$

$$\leq (|c_1| + |c_r|) \sum_{i=1}^{r} \mu(S(A_i, B_i)) < (|c_1| + |c_r|) \sum_{i=1}^{r} \frac{\varepsilon}{r \cdot (|c_1| + |c_r|)} = \varepsilon. \quad \square$$

EGOROFF'S THEOREM

1. Let E be a measurable set (finite measure), and f_n a sequence of measurable functions defined on E such that, for each $x \in E$, $f_n(x) \longrightarrow f(x)$, where f is a real-valued function. Then show that given any $\varepsilon, \delta > 0$ there exists a measurable set $A \subseteq E$ with $\mu(A) < \delta$ and an integer N such that, for all $x \notin A$, and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Proof. Let's define (as the hint suggested), for the fixed $\varepsilon > 0$ given,

$$G_n = \{x \in E; |f_n(x) - f(x)| \geq \varepsilon\}.$$

Then, because $f_n \longrightarrow f$ pointwise, we know $\{G_n; \text{i.o.}\} = \emptyset$. Thus,

$$0 = \mu(\emptyset) = \mu(\{G_n; \text{i.o.}\}) = \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} G_k\right) = \lim_{n \longrightarrow \infty} \mu\left(\bigcup_{k=n}^{\infty} G_k\right).$$

The last equality is an application of problem 1.4.6a, since the unions are obviously nested, and the total space E has finite measure. Thus, for the fixed $\delta > 0$ given, we can choose some N such that

$$\mu\left(\bigcup_{k=N}^{\infty} G_k\right) < \delta.$$

Then if we let

$$A = \bigcup_{k=N}^{\infty} G_k,$$

we have $\mu(A) < \delta$, and for all $x \notin A$, $n \geq N$, $x \notin G_n$ so $|f_n(x) - f(x)| < \varepsilon$. □

2. Give an example that shows the assumption $\mu(E) < \infty$ is necessary in the above result.

So to find a counterexample to the proposition when $\mu(E) = \infty$, look to where we used the fact that is had finite measure in the proof. We needed it to say that the limit of $\mu(\cup G_k) \longrightarrow 0$, given that the intersection of the nested sets, $\{G_n; \text{i.o.}\}$, is empty. If these sets are allowed to have infinite measure, it is easy to come up with examples of a nested sequence of sets, all with infinite measure, that have empty intersection. One such example is:

$$A_n = (n, \infty), \text{ so } A_n \supset A_{n+1} \text{ and } \bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Now we construct the f_n so that the $G_n = (n, \infty)$. Let $E = \mathbb{R}$ and

$$f_n = \begin{cases} 1 & \text{if } x > n \\ 0 & \text{if } x \leq n. \end{cases}$$

Then it is clear that $f_n(x) \longrightarrow 0$ for all $x \in \mathbb{R}$, but if $\varepsilon = \frac{1}{2}$, for any N, the set

$$G_N = x \in \mathbb{R}; |f_N(x) - 0| \geq \frac{1}{2} = (N, \infty)$$

has infinite measure.

3. Let E be a measurable set (finite measure), and f_n a sequence of measurable functions defined on E such that, for each $x \in E$, $f_n(x) \longrightarrow f(x)$, where f is a real-valued function. Then given any $\eta > 0$, there exists a measurable set $A \subseteq E$ with $\mu(A) < \eta$ such that f_n converges uniformly to f on $E \setminus A$.

   Proof. For each $n \in \mathbb{N}$, let $\varepsilon_n = \frac{1}{n}$ and $\delta_n = 2^{-n}\eta$. Then from problem 1 above, we get $A_n \subseteq E$ with $\mu(A_n) < \delta_n = 2^{-n}\eta$ and some N_n such that for all $m \geq N_n$, $x \notin A_n$, $|f_m(x) - f(x)| < \varepsilon = \frac{1}{n}$. Then we let

   $$A = \bigcup_{n=1}^{\infty} A_n.$$

   This gives

   $$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n) < \sum_{n=1}^{\infty} 2^{-n}\eta = \eta$$

   and for any $\varepsilon > 0$, we can choose some $M > \varepsilon^{-1}$, and so $\forall m > N_M, \forall x \notin A$,

   $$x \notin A_m \Longrightarrow |f_m(x) - f(x)| < \frac{1}{M} < \varepsilon.$$

   □