18.103 Problem Set 3 Partial Solutions

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1.4.18 We have that Chebyshev's inequality implies

$$\mu\left(\left\{\omega; \left|\frac{S_n(\omega)}{n}\right| < \varepsilon\right\}\right) \leq 3n^{-2}\varepsilon^{-4}.$$

Now we are meant to choose a sequence $\varepsilon_n \longrightarrow 0$ such that

$$\sum_{n=1}^{\infty} n^{-2}\varepsilon_n^{-4}$$

is finite. Here, we have to be explicit, we need to actually write down a formula for ε_n satisfying these criteria to exhibit that such a sequence exists. It's good to be able to rattle off a few convergent infinite sums for just this sort of occassion (or for dinner parties); the two most common examples are

$$\sum_{n=1}^{\infty} n^{-p} \text{ for } p \in \mathbb{R}, p > 1, \text{ and } \sum_{n=1}^{\infty} a^n \text{ for } a \in \mathbb{R}, |a| < 1.$$

(These two examples can be proven by the integral test and the geometric series formula respectively.)

So we'd like to choose our ε_n so that the summand in the infinite sum above matches one of these sums that is known to converge. Because we are multiplying a power of ε_n by n^{-2}, getting the sum to look like the first sum seems much more promising. So in other words, we'd like to choose ε_n so that both $\varepsilon_n \longrightarrow 0$ and $\forall n \in \mathbb{N}$,

$$n^{-2}\varepsilon_n^{-4} = n^{-p}$$

for some $p > 1$. So we solve for ε_n:

$$\begin{aligned} n^{-2}\varepsilon_n^{-4} &= n^{-p} \\ \varepsilon_n^{-4} &= n^{2-p} \\ \varepsilon_n^{4} &= n^{p-2} \\ \varepsilon_n &= n^{\frac{p-2}{4}} \end{aligned}$$

Now we have to choose p subject to two restraints: for our sum to converge, we need $p > 1$, and for $\varepsilon_n \longrightarrow 0$, we need $\dfrac{p-2}{4} < 0 \Longrightarrow p < 2$. So let $p = \frac{3}{2}$. Then

$$\varepsilon_n = n^{\frac{\frac{3}{2}-2}{4}} = n^{-\frac{1}{8}}.$$

So now we define the set

$$A_n = \left\{\omega; \left|\frac{S_n(\omega)}{n}\right| > n^{-\frac{1}{8}}\right\} \in \mathcal{F}.$$

Then we have (from the first equation above),

$$\mu(A_n) \leq 3n^{-2}(n^{-\frac{1}{8}})^{-4} = 3n^{-\frac{3}{2}}.$$

Thus,

$$\sum_{n=1}^{\infty} \mu(A_n) \le \sum_{n=1}^{\infty} 3n^{-\frac{3}{2}} = 3\sum_{n=1}^{\infty} n^{-\frac{3}{2}} < \infty.$$

So by the first Borel-Cantelli lemma, if $A = \{A_i; \text{i.o.}\}$, then $\mu(A) = 0$. Now it isn't too hard to derive the law of large numbers.

$$0 = \mu(A) = \mu(\{A_i; \text{i.o.}\}) = \mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) = \lim_{n\longrightarrow\infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) =$$

$$\lim_{n\longrightarrow\infty} \mu\left(\bigcup_{k=n}^{\infty}\left\{\omega; \left|\frac{S_k(\omega)}{k}\right| > k^{-\frac{1}{8}}\right\}\right) = \lim_{n\longrightarrow\infty} \mu\left(\left\{\omega; \exists k \ge n \text{ s.t. } \left|\frac{S_k(\omega)}{k}\right| > k^{-\frac{1}{8}}\right\}\right)$$

$$\ge \mu\left(\left\{\omega; \left|\frac{S_i(\omega)}{i}\right| \not\to 0\right\}\right),$$

since

$$\left\{\omega; \left|\frac{S_i(\omega)}{i}\right| \not\to 0\right\} \subseteq \left\{\omega; \exists k \ge n \text{ s.t. } \left|\frac{S_k(\omega)}{k}\right| > k^{-\frac{1}{8}}\right\} \quad \forall n \in \mathbb{N}.$$

So we have shown

$$\mu\left(\left\{\omega; \left|\frac{S_i(\omega)}{i}\right| \not\to 0\right\}\right) = 0,$$

the law of large numbers.

1.4.19 We have $X = \{x_1, x_2, \ldots\}$, and $P_i \in \mathbb{R}$ nonnegative with

$$\sum_{i=1}^{\infty} P_i = 1 \text{ and } \mu(A) = \sum_{x_i \in A} P_i.$$

We prove this by contradiction: assume there does exist an infinite sequence $\{A_i\}_{i=1}^{\infty}$ of independent sets, all with measure $\frac{1}{2}$. Then let $x_r \in X$, $n \in \mathbb{N}$. For each A_i, $i \in \{1, 2, \ldots, n\}$, we either have $x_r \in A_i$ or $x_r \notin A_i$. If $x_r \in A_i$, let $B_i = A_i$, if $x_r \notin A_i$, let $B_i = A_i^c$. Then for each i, $x_r \in B_i$, and either $\mu(B_i) = \mu(A_i) = \frac{1}{2}$ or $\mu(B_i) = \mu(A_i^c) = 1 - \mu(A_i) = 1 - \frac{1}{2} = \frac{1}{2}$. So $\mu(B_i) = \frac{1}{2}$ for all i.

Since the $A_1, A_2, \ldots, A_n$ are independent, by problem 1.4.10a (applied as necessary), the $B_1, B_2, \ldots, B_n$ are independent. Thus,

$$\mu\left(\bigcap_{i=1}^{n} B_i\right) = \prod_{i=1}^{n} \mu(B_i) = \frac{1}{2^n}.$$

But $x_r \in B_i$ for each i, so $x \in \bigcap B_i$, so

$$\frac{1}{2^n} = \mu\left(\bigcap_{i=1}^{n} B_i\right) = \sum_{x_j \in \cap B_i} P_j \ge P_r.$$

So we have shown that $P_r \leq \frac{1}{2^n}$. Since n was arbitrary, we can let $n \longrightarrow \infty$. Then $P_r = 0$. But r was also arbitrary, so $P_i = 0$ for all i. But this blatantly contradicts

$$\sum_{i=1}^{\infty} P_i = 1,$$

so our assumption must be false. Therefore, there does not exist an infinite sequence of independent sets with measure $\frac{1}{2}$.