18.103 Problem Set 2 Partial Solutions

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1.3.10 We have to prove 2^X is a complete metric space, when it has the distance function $d(A, B) = \mu^*[S(A, B)]$. Completeness is the property that all Cauchy sequences converge to an element of the space, so we must take a generic Cauchy sequence in 2^X, say (A_i), and prove that it converges to some $A \in 2^X$.

Usually, for these completeness arguments, you want to come up with a good candidate for the limit, and then prove that the sequence converges to it. So here, we want a subset of X that contain exactly the elements that are "eventually" in the all of the sets A_i. Thus, let's try the set

$$A = \bigcup_{n=1}^{\infty} \left[\bigcap_{i=n}^{\infty} A_i \right].$$

This is the natural choice, because $n = 1$ gives the points that are in all the $A_i's$, and as n gets larger we ignore more and more of the first $A_i's$, since we are more interested in the tail of the sequence to find its limit. However, this is slightly too restrictive a set. Consider the sequence of sets:

$$A_n = [0, 1] \setminus \{x \in [0, 1] | \exists m \in \mathbb{Z} \text{ with } H_n < x + m < H_{n+1}\},$$
$$\text{where } H_n = \sum_{j=1}^{n} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}.$$

(see associated picture)

You are probably aware that the harmonic series diverges, that is, $H_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Thus, given $x \in [0, 1]$, for any $N \in \mathbb{N}$, there is some $n > N$ with $H_n \le x + N < H_{n+1}$, so $x \notin A_n$. Therefore $x \notin \cap_{i=N}^{\infty} A_i$ for any $N \in \mathbb{N}$, so $x \notin A$, so $A = \emptyset$. However, under the metric given, it is easy to see that $d(A_n, [0, 1]) = \frac{1}{n+1} \longrightarrow 0$, so the sequence actually converges to $[0, 1]$.

To make the set not quite as restrictive, we will pick a subsequence of (A_n) and use the above formula with the subsequence. We may then apply the fact that if a subsequence of a Cauchy sequence converges to a limit, the entire sequence converges to the same limit. We will choose the subsequence to make the future calculations easy. Since (A_n) is Cauchy, for any $j \in \mathbb{N}$, $\exists N_j \in \mathbb{N}$ such that $\forall m, n \ge N_j$,

$$\frac{1}{2^j} > d(A_m, A_n) = \mu^*[S(A_m, A_n)].$$

So our subsequence is $(A_{N_j})_{j=1}^{\infty} = (B_j)$, and we let

$$B = \bigcup_{n=1}^{\infty} \left[\bigcap_{j=n}^{\infty} B_j \right].$$

Now, given $\varepsilon > 0$, we show that for all $m \geq M$, for $2^{2-M} < \varepsilon$, $d(B_m, B) < \varepsilon$. We have

$$\begin{aligned} d(B_m, B) &= \mu^*[S(B_m, B)] = \mu^*[(B_m \setminus B) \cup (B \setminus B_m)] \leq \mu^*[B_m \setminus B] + \mu^*[B \setminus B_m] \\ &= \mu^*[B_m \setminus \bigcup_{n=1}^{\infty} (\bigcap_{i=n}^{\infty} B_i)] + \mu^*[(\bigcup_{n=1}^{\infty} (\bigcap_{i=n}^{\infty} B_i)) \setminus B_m] \\ &= \mu^*[\bigcap_{n=1}^{\infty} (B_m \setminus \bigcap_{i=n}^{\infty} B_i)] + \mu^*[\bigcup_{n=1}^{\infty} ((\bigcap_{i=n}^{\infty} B_i) \setminus B_m)] \\ &= \mu^*[\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (B_m \setminus B_i)] + \mu^*[\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (B_i \setminus B_m)] \end{aligned}$$

Now each summand must be controlled. The first summand is the measure of an intersection of nested sets, since

$$\bigcup_{i=n}^{\infty} (B_m \setminus B_i) \supseteq \bigcup_{i=n+1}^{\infty} (B_m \setminus B_i).$$

Therefore it can be bounded above by the measure of one of its terms:

$$\begin{aligned} \mu^*[\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (B_m \setminus B_i)] &\leq \mu^*[\bigcup_{i=M}^{\infty} (B_m \setminus B_i)] \leq \mu^*[(B_m \setminus B_M) \cup \bigcup_{i=M}^{\infty} (B_{i+1} \setminus B_i)] \\ &\leq \mu^*[(B_m \setminus B_M)] + \sum_{i=M}^{\infty} \mu^*[(B_{i+1} \setminus B_i)] < 2^{-M} + \sum_{i=M}^{\infty} 2^{-i} = 2^{-M} + 2^{1-M} = 3 \cdot 2^{-M}. \end{aligned}$$

The second summand is the measure of a union of nested sets, since

$$\bigcap_{i=n}^{\infty} (B_i \setminus B_m) \subseteq \bigcap_{i=n+1}^{\infty} (B_i \setminus B_m).$$

So its measure is equal to the limit of the measures of the sets in the nested sequence:

$$\mu^*[\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (B_i \setminus B_m)] = \lim_{n \longrightarrow \infty} \mu^*[\bigcap_{i=n}^{\infty} (B_i \setminus B_m)] \leq \limsup_{n \longrightarrow \infty} \mu^*[B_n \setminus B_m] \leq 2^{-M}.$$

So, we finally get:

$$d(B_m, B) \leq \mu^*[\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (B_m \setminus B_i)] + \mu^*[\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (B_i \setminus B_m)] \leq 3 \cdot 2^{-M} + 2^{-M} = 2^{2-M} < \varepsilon.$$

Which shows that the subsequence, and thus the sequence, converges to B, so 2^X is complete.