18.103 Problem Set 1

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PROBLEM 1.1.21

This problem was to show that intervals with nonzero length aren't, quite unsurprisingly, sets of Lebesgue measure zero. So we consider an interval $[a, b]$ with $a < b$. If this was a set of measure zero, for any $\varepsilon > 0$, there would be a cover of this set by open intervals of total length less than ε.

So, more mathematically, we assume $[a, b]$ has Lebesgue measure zero. Therefore, for $\varepsilon = \dfrac{b-a}{2}$, $\exists \{I_j\}$, a (possibly countably infinite) collection of open intervals (s_j, t_j) such that

$$[a, b] \subseteq \bigcup_j I_j \text{ and } \sum_j (t_j - s_j) < \varepsilon = \frac{b-a}{2}.$$

Thus $\{I_j\}$ is an open cover of $[a, b]$, a compact set by Heine-Borel. So there is a finite subcover, some $\{I_k\}_{k=1}^N$, which we can assume is chosen so no interval is contained in another. Order these intervals by the size of s_k, so $s_1 \leq s_2 \leq ....$ Since $a \in \cup_k I_k$, there is some greatest k with $a \in I_k$, that is, $s_k < a < t_k$, and some least k' with $b \in I_{k'}$. Then since these intervals aren't contained within each other and cover $[a, b]$, we have

$$s_k < a < s_{k+1} < t_k < s_{k+2} < t_{k+1} < \cdots < s_i < t_{i-1} < s_{i+1} < t_i < \cdots < s_{k'} < t_{k'-1} < b < t_{k'}.$$

So, we can calculate:

$$\sum_{i=k}^{k'} \big(t_i - s_i\big) > (t_k - s_k) + \sum_{i=k+1}^{k'} \big(t_i - t_{i-1}\big) = (t_k - s_k) + (t_{k'} - t_k) = t_{k'} - s_k > b - a.$$

But we also have:

$$\sum_{i=k}^{k'} \big(t_i - s_i\big) < \sum_j (t_j - s_j) < \varepsilon = \frac{b-a}{2},$$

So we have shown $b - a < \dfrac{b-a}{2}$, a contradiction. Therefore our assumption is false, so $[a, b]$ does not have Lebesgue measure zero.