Commutative and noncommutative symplectic resolution and perverse sheaves

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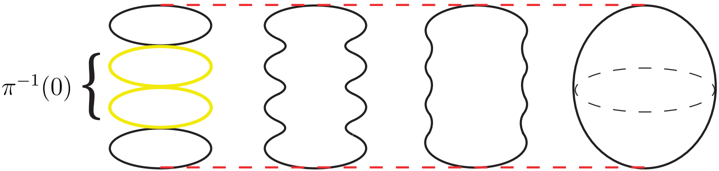
Symplectic resolutions.

Let $\pi : X \to Y$ be a *symplectic resolution*.
So X, Y are algebraic varieties over a field k (for now $k = \mathbb{C}$), π is a resolution of singularities and X has an algebraic symplectic form ω. We assume that Y is normal and that the multiplicative group $\mathbb{G}_m$ acts on X, Y contracting Y to a point 0 and $t^*(\omega) = t^2\omega$.
Examples: 1) $X = T^*(G/B)$ is the cotangent of the flag variety for a semisimple algebraic group G; $Y = \mathcal{N} \subset \mathfrak{g}$ is the set of nilpotent matrices.
Or more generally $Y \subset \mathcal{N}$ is a transversal slice to an orbit and $X \subset T^*(G/B)$ is the preimage of Y.

2) $X = M(w, v)_\theta$, $Y = M_0(w, v)$ are the Nakajima *quiver varieties.*
Thus $M(w, v)_\theta$ is the moduli space of representations of a Nakajima quiver which are stable with respect to stability θ and $M_0(w, v)$ is the affine coarse moduli space.
3) $Y = V/\Gamma$ where V is a symplectic vector space.

Kleinian example

For example, $Y = \mathbb{A}^2/\Gamma$ for a finite subgroup $\Gamma \subset SL(2)$ is a special case of all three types of examples.



$X \to Y = \{xy = z^3\}$ (fibered over $\{z\}$)

K-equivalence $\Rightarrow$ D-equivalence for symplectic resolutions.

Given X, Y we may also have another symplectic resolution $\pi' : X' \to Y$. For an algebraic variety X abbreviate $D^b(Coh(X))$ to $D(X)$.

Conjecture. (a special case Kawamata "*K-equivalence implies D-equivalence*" conjecture (2002)) *Let X, X' be two resolutions of singularities of the same singular space Y. Assume that the canonical line bundles K_X, $K_{X'}$ are trivial. Then there exists an equivalence $D^b(Coh(X)) \cong D^b(Coh(X'))$.*

Theorem. [R.B. and D. Kaledin for $Y = V/\Gamma$ (2004); Kaledin in general (2005)].
Conjecture holds when $X \to Y$, $X' \to Y$ are symplectic resolutions.
Remark. The conjecture does not say *how canonical* this equivalence is supposed to be. We will state a more precise conjecture in our case.

Equivalences from paths in the space of Kahler parameters

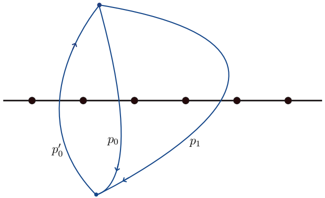
Notice: $Pic(X) = Pic(X')$ (this is an easy fact since $X \supset U \cong U' \subset X'$ where U and U' have complement of codimension at least two).
Let $\Lambda = Pic(X)/\text{torsion}$. It is known that Λ is a lattice, $\Lambda = H^2(X, \mathbb{Z})/\text{torsion}$ which is thus canonically defined given Y (assuming a symplectic resolution X exists).

Set $V = \Lambda \otimes \mathbb{R} = H^2(X, \mathbb{R})$. We will define an open subset $V^0_{\mathbb{C}} \subset V_{\mathbb{C}}$ (complement to some hyperplanes) and propose that the equivalence depends on the homotopy class of a path connecting two regions in V^0.
Remark. This implies that $\pi_1(V^0_{\mathbb{C}})$ acts on $D(X)$, so we should have a *local system* of categories on V_0. We expect (and can prove in some cases) it comes from a richer structure related to Bridgeland variations of stabilities.

Example: cotangent to projective spaces

$X = T^*(\mathbb{P})$, $X' = T^*(\check{\mathbb{P}})$ where $\mathbb{P} = \mathbb{P}^{n-1}$ is a projective space and $\check{\mathbb{P}}$ is the dual projective space; Y is the space of nilpotent $n \times n$ matrices of rank at most one.

Then $V = \mathbb{C}$, $V^0 = \mathbb{C} \setminus \mathbb{Z}$. The equivalence will depend on a path from the upper half plan to the lower half plane. The equivalence corresponding to the path around p_0 is given by the correspondence $X \times_Y X'$, the one corresponding to p_0' by the correspondence $X' \times_Y X$. Using the loop p_1 will get the equivalence given be the sheaf $pr_1^*(\mathcal{O}(2)) \otimes pr_2^*(\mathcal{O}(-2))$ on $X \times_Y X'$.



Equivalences via quantization in positive characteristic

The idea of proof of the Theorem is based on *quantization in positive characteristic*. This is done in five steps:

I) Replace X, X', Y by "the same" varieties X_k, X'_k, Y_k over k of characteristic $p \gg 0$.
II) Find compatible quantizations X_k, X'_k, Y_k.
III) Prove derived quantized affineness:

$$D^b(X_k^{quant} - mod) \cong D^b(Y_k^{quant} - mod) \cong D^b((X')_k^{quant} - mod).$$

IV) Use the *p-center* phenomenon to relate modules over quantized positive characteristic varieties to coherent sheaves.
V) Lift everything to characteristic zero.

Some examples

Example of the p-center phenomenon: x^p, y^p are in the center of

$$k\langle x, y\rangle / xy - yx = 1.$$

Example of the method: $X = T^*(\mathbb{P}^1)$, quantization produces the sheaf of twisted differential operators on $\mathbb{P}^1$. Its restriction to zero section is $Fr_*(\mathcal{O}(i))$.
For $-1 < i < p-1$ we have $Fr_*(\mathcal{O}(i)) = \mathcal{O}^{i+1} \oplus \mathcal{O}(-1)^{p-i-1}$. Only indecomposable summands, not their multiplicities matter, so lifting to characteristic zero get $\mathcal{O} \oplus \mathcal{O}(-1)$ on $T^*(\mathbb{P}^1)$.

Local system of categories (and more!) from quantization in positive characteristic

Key point: which quantization do we use in step II? Different choices lead to different equivalences!

Program: study all possible quantizations and equivalences they yield.

Motivations:

A) Get a rich structure on $D(X)$ with apparent connections to mirror symmetry, quantum cohomology etc.

A$'$) Sometimes they are connected to local geometric Langlands duality.

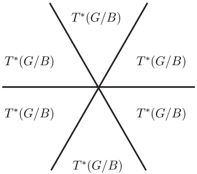
B) The quantized algebras $\mathcal{O}(Y^{quant})$ are oftentimes of interest in representation theory.

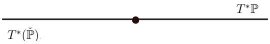
Applications to representation theory

In the setting of Example 1 these are quotients of *enveloping algebras* of semi-simple Lie algebras or finite W-algebra; in Example 3 get rational Cherednik algebras (rDAHA) or symplectic reflection algebras (SRA).
Hope to develop new generalizations of *Kazhan-Lusztig theory*: relate irreducible representations in positive characteristic to *canonical basis* in the Grothendieck ring $K^0(D(X)) \otimes \mathbb{Q} \cong H^*(X, \mathbb{Q})$.
[The last isomorphism follows from De Concini-Lusztig-Procesi in Example 1, Kaledin in general].
Existing applications: count of the number of finite dimensional representations of wreath product symplectic reflection algebras (Etingof's conjectures, joint with Losev).

Geometric picture: symplectic resolutions depending on a chamber

The following is standard in Example 1, due to Nakajima in Example 2 and to Namikawa in general.
$V_{\mathbb{R}}$ is partitioned into rational cones, and a resolution X_C can be attached to a cone C. There is a group W (the Weyl group) acting on V and for $\lambda \in C$ we have $w(\lambda)$ is *ample* on X_C for some w.
In Example 1 we get just one resolution and the usual stratification by Weyl chambers.




$T^*\mathbb{P}$
$T^*(\check{\mathbb{P}})$

Noncommutative resolutions depending on alcoves

In the last slide we saw that symplectic resolutions of a given singular space depend on a cone (chamber) in the space $V_{\mathbb{R}}$. We will now look at some shifts of the hyperplanes which partition $V_{\mathbb{R}}$ into compact pieces called *alcoves*, and describe *noncommutative* resolutions parametrized by those alcoves.
We also let $V_{\mathbb{C}}^0$ denote the complement in $V_{\mathbb{C}}$ to the complexification of the hyperplanes.
Can define a quantization depending on $\lambda \in \Lambda = Pic(X)$, getting $\mathcal{O}_X^{quant}(\lambda)$ etc.
E.g.. for $X = T^*(G/B)$ it's the sheaf of *twisted differential operators* on G/B.

Noncommutative resolution via quantization in positive characteristic

The Theorem part in the next statement is joint with R. Anno, R.B., I. Mirkovic, based on joint work with Riche, Mirkovic; Mirkovic and Rumynin.

Claim. (Theorem in Example 1 (assuming $H^2(X) \cong H^2(T^*(G/B))$) and conjecture in general)

There exists a collection of affine hyperplanes which partition $V_{\mathbb{R}}$ into alcoves, so that

1) The equivalence obtained by the procedure above works when $\frac{\lambda}{p}$ is in one of the alcoves.

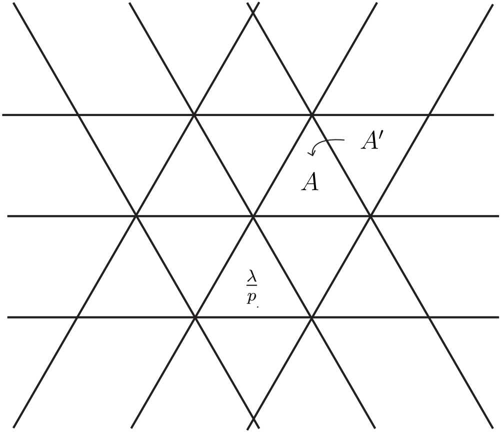
2) For $\frac{\lambda}{p}$, $\frac{\mu}{p}$ in the same alcove we get the same equivalence.

3) Moreover, the algebras obtained by quantizing Y in positive characteristic can be lifted to characteristic zero resulting in algebras (well defined up to a Morita equivalence) $\mathcal{A}_A$ depending on an alcove A. They come with an equivalence $D(\mathcal{A}_A - mod) \cong D(X)$.

Equivalences via quantization in positive characteristic

4) All these equivalences fit into a representation of the Poincare groupoid of $V^0_{\mathbb{C}}$.
Here: to a point in an alcove we assign the category $D^b(\mathcal{A}_A - mod)$.
To a point in the region $V_{\mathbb{R}} + iC$ *we assign* $D(X_C)$.
To a straight line connecting $x \in A$ *to* $y \in V_{\mathbb{R}} + iC$ *we assign the canonical equivalence* $D^b(\mathcal{A}_A - mod) \cong D(X)$ *above.*
5) For two adjacent alcoves the equivalence $D^b(\mathcal{A}_A - mod) \cong D^b(\mathcal{A}_{A'} - mod)$ *corresponding to the counterclockwise path around the separating hyperplane is a* perverse equivalence *governed by the central charge map, similar to Bridgeland's stability axioms.*

Question: Kapranov and Schechtman defined perverse sheaves of categories (*perverse Schobers*). Does the above extend to a perverse Schober on V?
Easy (but important) remark: The action of $\pi_1(V^0)$ on $D(X)$ lifts to $D^b(Coh^{\mathbb{G}_m}(X))$.


A'
A
$\frac{\lambda}{p}$

Relation to quantum cohomology

Claim. (Theorem in Example 1 (if $H^2(X) = H^2(G/B)$) based on a result of Braverman, Maulik and Okounkov, Conjecture in general).
The action of $\pi_1(V^0_{\mathbb{C}})$ on $K^0(Coh^{\mathbb{G}_m}(X))$ is isomorphic to the monodromy of equivariant quantum connection $QH^*_{\mathbb{C}^*}(X)$.

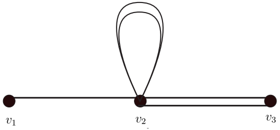
This confirms the idea that quantum cohomology should be related to Bridgeland stabilities proposed by Bridgeland based on mirror symmetry.

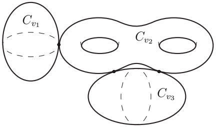
A precise relation of the Claim to mirror symmetry has not yet been completely worked out.

From quiver varieties to perverse sheaves

To describe the basis of irreducibles in $K^0(\mathcal{A}_A - mod)$ one wants to apply the pattern of Kazhdan-Lusztig theory and realize it as a category of *perverse sheaves*.
A result from a joint project with M. Kapranov suggests a way to do this in the case of a quiver variety.

Start with a quiver Q, we allow loops and other multiple edges.
Build a compact complex curve C with nodal singularities as follows.
The components of C are in bijection with vertices of Q. For a vertex v the component C_v has genus g.
The curve C is obtained from the disjoint union of C_v by choosing m points on C_v and gluing each one of them to a point on $C_{v'}$ for any pair of vertices v, v' connected by an edge of multiplicity m.
We also trivialize the tensor product of tangent lines at each pair of glued points.


v_1
v_2
v_3


C_{v_1}
C_{v_2}
C_{v_3}

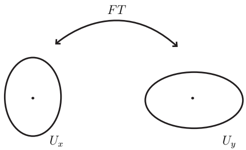
Microlocal sheaves

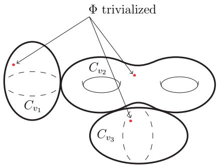
Definition. A *microlocal sheaf* on C is a perverse sheaf $\mathcal{F}$ on the normalization $\tilde{C}$ together with an identification

$$FT(\mathcal{F}|_{U_x}) \cong \mathcal{F}|_{U_y}.$$

Here $x, y \in \tilde{C}$ are two points projecting to the same point on C, U_x, U_y are small disks around x, y and FT is the local Fourier transform.

A *marked microlocal sheaf* is a microlocal sheaf s.t. the corresponding sheaf on $\tilde{C}$ has singularities at the preimages of the double points plus one additional marked point on each component, together with trivialization of the vanishing cycles at those marked points.


FT
U_x
U_y


Φ trivialized
C_{v_1}
C_{v_2}
C_{v_3}

Microlocal sheaves and multiplicative quiver varieties

Theorem. (R.B. and M. Kapranov) *A multiplicative quiver variety is the moduli space of marked microlocal sheaves.*

Remark. A closely related construction is due to Crawley-Boevey and a related construction to Yamakawa.

Quiver varieties and perverse sheaves

The following Conjecture is motivated by the standard fact that the moduli space (stack) $LocSys_{DR}$ of De Rham local systems on a smooth curve is the cotangent bundle to the stack Bun_n, the moduli stack of rank n vector bundles.

The second part of the conjecture is motivated by analogy with geometric Langlands duality.

Conjecture. *The moduli space of De Rham microlocal sheaves is the twisted cotangent bundle of a moduli space of object (vector bundles on C with an additional structure) that we call marked vector bundles.*
$\mathcal{A}_A - mod$ *is equivalent to a full subcategory in the category of perverse sheaves on the space of marked vector bundles.*

Remark. Something related to our marked vector bundles was defined by Crawley-Boevey who called them vector bundle representations of the quiver.