Canonical bases and coherent sheaves.

Canonical bases and coherent sheaves.

1) Kazhdan-Lusztig theory.

Recall: $\mathfrak{g}$ - semisimple Lie algebra / $\mathbb{C}$.

$M_\lambda = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$ - Verma module,

$M_\lambda \twoheadrightarrow L_\lambda$ - irreducible quotient.

Problem: compute characters of L_λ.

Solution (KL et al.). Consider the category generated by L_λ (category $\mathcal{O}$).

Focus on principal block, $\mathcal{O}_0$: irreducibles $L_{w\cdot 0}$, Cartan action diagonalizable.

$$K(\mathcal{O}_0) \simeq \mathbb{Z}[W]$$

$$[M_{w\cdot(-2\rho)}] \longmapsto w.$$

Then $[L_{w\cdot(-2\rho)}] \longmapsto C_w$ — canonical basis.

[KL conjectures proved 1980 by Beilinson–Bernstein and Brylinski–Kashiwara].

Key features.

– relation to geometry

Localization Theorem: $\mathcal{O}_0 \simeq D\text{-mod}_B(G/U) \simeq$

$$\simeq \mathrm{Perv}_B(G/U).$$

– HC bimodules description

$$D\text{-mod}_B(G/U) \simeq D\text{-mod}_G(G/B \times G/U) \simeq$$

$$\simeq (\mathfrak{g} \oplus \mathfrak{g}, G)\text{-mod}^{0,\hat{0}}.$$

– <u>Symmetries.</u> 1) $D^b(\mathcal{O}_0) \circlearrowleft \mathbb{B} \times \mathbb{B}$

$\mathbb{B}$ – Artin <u>braid group</u>.

Comes from correspondences

$$\begin{array}{ccc} & (G/B)^2_w & \\ \swarrow & & \searrow \\ G/B & & G/B \end{array}$$

cd

- Symmetries.

i) $D^b(\mathcal{O}_0) \circlearrowleft B \times B$

B - Artin braid group.

Comes from correspondences

$$\begin{tikzcd} & (G/B)^2_w \arrow[dl] \arrow[dr] & \\ G/B & & G/B \end{tikzcd}$$

ii) Wall crossing functor

$\Xi_\alpha = T_{\mu \to 0} \circ T_{0 \to \mu}$

μ on the α wall

T - translation functor.

$\Xi_\alpha : \mathcal{O}_0 \longrightarrow \mathcal{O}_0$ - exact self-adjoint.

sends projective to projective.

iii) Zuckerman functors

$Z_\alpha : D^b(\mathcal{O}_0) \longrightarrow D^b(\mathcal{O}_0)$

comes from $\pi_\alpha^* \pi_{\alpha *}$, $\pi_\alpha : G/B \xrightarrow[\mathbb{P}^1]{} G/P_\alpha$

Z_α - semisimple (deep fact $\Rightarrow$ KL conj.).

NB: irreducibles $L_{w\cdot 0}$ are (up to shift) summands of $Z_{\alpha_1} \cdots Z_{\alpha_n}(L_{-2\rho})$, $L_{-2\rho} = M_{-2\rho}$

Indecomposable projectives are summands of $\Xi_{\alpha_1} \cdots \Xi_{\alpha_n}(M_0)$.

Their classes give canonical basis C_w and dual canonical basis C_w'.

Also summands of $\Xi_{d_1} \cdots \Xi_{d_n}(L_{-2\rho} = M_{-2\rho})$ are <u>tilting</u> objects, give twisted dual canonical basis.

- Koszul duality [Beilinson–Ginzburg–Soergel]
  is an autoequivalence

$$D^b(\mathcal{O}_0^{gr}) \simeq D^b(\mathcal{O}_0^{gr})$$

inter   i   irreducibles and projectives

(variant   irreducibles and tilting)

Coherent sheaves.

$\tilde{N} = T^*(G/B)$.

Canonical basis in $K(Coh^G(\tilde{N}))$.

controls characters of tilting modules for quantum groups at $= \sqrt[\ell]{1}$, also the objects representing elements of the basis contain information about $H^*(b_q, T_\lambda)$, $H^*(u_q, T_\lambda)$,

T_λ -tilting, b_q - Borel in the small quantum group u_q.

Also controls characters of some $\hat{\mathfrak{g}}$ -modules.

$D^b(\mathrm{Coh}^G(\tilde{N})) \circlearrowleft B_{aff}$ – extended affine braid group.

$B_{aff} = \langle \tilde{s}_\alpha, \theta_\lambda \mid \lambda \in \Lambda, \alpha \in \Sigma \rangle$

Λ – weight lattice

Σ – simple roots.

$\theta_\lambda : \mathcal{F} \longmapsto \mathcal{F} \otimes \mathcal{O}(\lambda)$.

$\tilde{s}_\alpha$ comes from

$$\begin{tikzcd} & X_\alpha \arrow[dl] \arrow[dr] & \\ \tilde{N} & & \tilde{N} \end{tikzcd}$$

$X_\alpha = \Delta \cup N_\alpha$, Δ – diagonal,

(up to $\mathcal{O}(-\rho)$ twist) N_α – conormal to $(G/B)^2_{s_\alpha}$

Z_α , $\alpha \in \Sigma_{aff}$.

If $\alpha \in \Sigma$, then (up to a $(-\rho)$ twist) given by

$$\begin{tikzcd} & N_\alpha \arrow[dl] \arrow[dr] & \\ \tilde{\mathcal{N}} & & \tilde{\mathcal{N}} \end{tikzcd}$$

For $\alpha_0 \in \Sigma_{aff} \setminus \Sigma$, $Z_{\alpha_0} = b\, Z_\alpha\, b^{-1}$

where $\alpha \in \Sigma$, $b\, \tilde{s}_\alpha\, b^{-1} = \tilde{s}_{\alpha_0}$.

<u>Thm.</u> a) Classes of indecomposable summands of $Z_{\alpha_1} \dots Z_{\alpha_n}(\mathcal{O})$ are (up to sign) elements of the <u>canonical basis</u> in $K(Coh^G(\tilde{\mathcal{N}}))$

b) These are (up to shifts) irreducibles in the heart of some t-structure.

<u>Remark</u> on the meaning of a).

$$K(\mathrm{Coh}^{G}(\tilde{\mathcal{N}})) \circlearrowleft W_{aff}$$

$$K(\mathrm{Coh}^{\mathbb{G}_m^{\vee} \times G}(\tilde{\mathcal{N}})) \circlearrowleft H_{aff} \text{ - affine Hecke algebra}$$

$$\begin{array}{c} \| \\ \boxed{H_{aff} \otimes_{H_{fin}} \mathrm{sgn}.} \\ \uparrow \\ H_{aff} \end{array}$$

$$\mathbb{Z}[W_{aff}] \twoheadrightarrow K(\mathrm{Coh}^G(\hat{\mathcal{N}}))$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \\ C_w & \longmapsto & 0 \text{ or element of canonical basis.} \end{array}$$

$w \in W_{aff}$

What about C_w itself?

Dual canonical basis C_w'?

Answer.

$\mathbb{Z}[W_{aff}] \simeq K\, Coh^G(\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}})$

Again have Z_α, B_{aff} acting on the left & right.

Direct summands of $Z_{\alpha_1} \cdots Z_{\alpha_n}(\mathcal{O}_\Delta)$

give canonical basis (up to sign).

These are irreducibles (up to shift) of some t-structure, related to category $\mathcal{O}$ for $\hat{\mathfrak{g}}$, also to HC bimodules in positive characteristic [with Riche in progress]

NB Need to consider $\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ as a <u>derived</u> scheme.

But also

$$\mathbb{Z}[W_{aff}] \simeq K\, Coh^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$$

$$D^b\, Coh^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \circlearrowleft \Xi_\alpha.$$

$$\tilde{\mathfrak{g}} \xrightarrow[\pi_\alpha]{} \tilde{\mathfrak{g}}_\alpha. \qquad \Xi_\alpha = \pi_\alpha^* \pi_{\alpha *}.$$

Summands of $\Xi_{\alpha_1} \cdots \Xi_{\alpha_n}(\mathcal{O}_\Delta)$ (tilting)

- dual canonical basis C'_w

Homotopy category of tilting $\xrightarrow{\sim} D^b\, Coh^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$

$Ho(\mathcal{T})$

$$\Updownarrow$$

$$Ext^{>0}(T_1, T_2) = 0$$

$$T_1, T_2 \in \mathcal{T}$$

$$\langle T_i \rangle = D^b\, Coh^G(\quad)$$

But also

$$\mathbb{Z}[W_{aff}] \simeq K\, Coh^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$$

$$D^b\, Coh^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \circlearrowleft \Xi_{\alpha}.$$

$$\tilde{\mathfrak{g}} \xrightarrow[\pi_\alpha]{} \tilde{\mathfrak{g}}_\alpha. \qquad \Xi_\alpha = \pi_\alpha^* \pi_{\alpha *}.$$

Summands of $\Xi_{\alpha_1} \cdots \Xi_{\alpha_n}(\mathcal{O}_\Delta)$ (tilting)

- dual canonical basis C_w'

Homotopy category of tilting $\xrightarrow{\sim} D^b\, Coh(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$

$Ho(\mathcal{T})$

$$D\, Coh^G(\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}) \longleftrightarrow D\, Coh^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$$

linear Koszul duality of Mirković–Riche.

$$C_w \longleftrightarrow C_w'.$$

Linear Koszul duality.

generalizes K D between

$\Lambda(V)$ and $\mathrm{Sym}(V^*)$

$\wr\!\wr$

$\mathrm{Tor}_\bullet^{\mathrm{Sym}(V)}(k, k)$

$\mathcal{E}$ – vector bundle. $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{E}$ - subbundles.

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ S \end{array}$$

$$D\,\mathrm{Coh}(\mathcal{E}_1 \times_{\mathcal{E}} \mathcal{E}_2) \longleftrightarrow D\,\mathrm{Coh}(\mathcal{E}_1^\perp \times_{\mathcal{E}_2^*} \mathcal{E}_2^\perp).$$

<u>Ex.</u> $S = (G/B)^2$, $\mathcal{E} \simeq \mathcal{E}^* \simeq \mathfrak{g} \times (G/B)^2$.

$\mathcal{E}_1 = \tilde{\mathcal{N}} \times_{(G/B)_1} (G/B)^2$, $\mathcal{E}_2 = \tilde{\mathcal{N}} \times_{(G/B)_2} (G/B)^2$

$\mathcal{E}_1^\perp = \tilde{\mathfrak{g}} \times_{(G/B)_1} (G/B)^2$, $\mathcal{E}_2^\perp = \tilde{\mathfrak{g}} \times_{(G/B)_2} (G/B)^2$

$$Z_\alpha \longleftrightarrow \Gamma_\alpha$$

Let now θ be an involution of G, $K = G^\theta$

We have

$$\tilde{\mathcal{N}} \times_{\mathfrak{g}} \mathfrak{k}^\perp \xleftrightarrow[\text{L.K.D.}]{} \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{k}$$

derived scheme (except if θ is split) i.e. $\mathfrak{t}^\theta = 0$ for a θ-invariant Cartan $\mathfrak{t}$

usual (non derived) scheme.

$D\,Coh^K(\tilde{\mathcal{N}} \times_{\mathfrak{g}} \mathfrak{k}^\perp) \circlearrowleft Z_\alpha$.

$D\,Coh^K(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{k}) \circlearrowleft \Xi_\alpha$

$\tilde{k}$

Conjecture (G-adjoint)

summands of $\Xi_{\alpha_1} \cdots \Xi_{\alpha_n}(\mathcal{O}_S) =: \mathcal{J}_K$

preimage of closed orbits

$$Ho(\mathcal{J}) \xrightarrow{\sim} D\,Coh^K(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{k})$$

classes of indecomposables - dual canonical basis

Rmk 1. In the $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ case proofs are based on geometric local Langlands – realization of the categories via constructible sheaves on $\mathcal{F}\ell$ – affine flag variety of the dual group.

In the $\tilde{k}$ setting such a realization is not known, although there're related conjectures by Ben-Zvi – Nadler.

Rmk 2. Conj. reduces to $\mathrm{Ext}^{>0}(T_1, T_2) = 0$
$T_1, T_2 \in \mathcal{J}_K$ & $\langle \mathcal{J}_K \rangle = D^b \mathrm{Coh}^G(\tilde{k})$.
Can prove vanishing for some $T_1, T_2 \in \mathcal{J}$.

Rmk 3. Category $\mathcal{J}$ should have a Soergel bimodule description

Recall Soergel bimodules is a full subcategory in

$$\mathrm{Coh}\Big(\mathfrak{t} \underset{\mathfrak{t}/W}{\times} \mathfrak{t}\Big)$$

$$\overset{\shortparallel}{\cup\, \Gamma_w}, \qquad \Gamma_w \subset \mathfrak{t} \times \mathfrak{t}$$

is the grap of w.

$$\mathfrak{t}/W \underset{\varkappa}{\hookrightarrow} \mathfrak{g}, \qquad \mathfrak{t} \underset{\mathfrak{t}/W}{\times} \mathfrak{t} \underset{\tilde{\varkappa}}{\subset} \tilde{\mathfrak{g}} \underset{\mathfrak{g}}{\times} \tilde{\mathfrak{g}}$$

Kostant slice.

In the $(\mathfrak{g}, K)$ setting $\tilde{\varkappa}_K : \mathfrak{t}_K \hookrightarrow \mathfrak{k}$

$$\mathfrak{t}_K \underset{\mathfrak{t}/W}{\times} \mathfrak{t} \subset \tilde{\mathfrak{k}} = \mathfrak{k} \underset{\mathfrak{g}}{\times} \tilde{\mathfrak{g}}$$

Affine Soergel bimodule S_{aff}.
[Abe] is a full subcategory
in $\{ (M, \gamma) \mid M \in Coh(\mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}), \quad \}$

γ - W_{aff} grading on
$M_F := M \otimes Frac(k[\mathfrak{t}])$
s.t. $Supp(M_F) \subset \Gamma_{\overline{w}}$.

Can define similarly $S_{aff}^K =$
$= \{ (M, \gamma) \mid M \in Coh(\mathfrak{t}_K \times_{\mathfrak{t}/W} \mathfrak{t}) \}$
γ ..

Can define a functor
$\sigma : J \longrightarrow S_{aff}$ - an upgrade of $\tilde{\mathcal{H}}^*$

$\sigma_K : J_K \longrightarrow S_{aff}^K$ - an upgrade of $\tilde{\mathcal{H}}^*_K$

σ is fully faithful, σ_K is expected to be fully faithful on J_K