April 30. Azumaya algebras

1. Definition.

~~Because assume to be noetherian~~

R-algebra A, f.g projective as module, is Azumaya if

(i) $A \otimes A^{op} \xrightarrow{\sim} \mathrm{End}_R(A)$

$\Updownarrow$

(ii) $\forall\ A \to k$ $\quad k \underset{R}{\otimes} A$ is central simple over k

$\Updownarrow$

(ii') $\forall\ A \to \bar{k}$ $\quad \bar{k} \underset{R}{\otimes} A$ is isomorphic to $M_n(\bar{k})$

[note: n is locally constant; constant $\Leftrightarrow$ A is fg projective of constant rank. $\mathrm{rk}_R A = n^2$]

Proof $\quad R \to S$ base change: $A_S = S \underset{R}{\otimes} A$

$A_S \underset{S}{\otimes} A_S^{op} \to \mathrm{End}_S(A_S)$ is obtained from (i) by tensoring with S.

Thus (i) iso $\Rightarrow$ iso for all S $\Rightarrow$ (ii). (ii) $\Leftrightarrow$ (ii') known.

Vice versa, let $A \otimes A^{op} \to \mathrm{End}(A) \to C \to 0$ be exact.

$C \neq 0 \Rightarrow \exists\ A \to k$ s.t. $C \underset{R}{\otimes} k \neq 0$ (take $p \in \mathrm{Supp}(C)$, $k = k(p)$).

2. Properties

2.1 $\quad F : \mathrm{mod}(R) \rightleftarrows \mathrm{mod}(A^e) : G$

$F(M) = A \otimes M$, $\quad G(N) = \mathrm{Hom}_{A^e}(A, N)$

is an equivalence of categories. (Morita eq.)

In particular, A is projective A_e-module (= separability)

~~$A \otimes$~~ $R = GF(R) = \mathrm{Hom}_{A^e}(A, A)$ the center

Claim A Azumaya iff it is central + separable.

2.2. A/R Azumaya, $R \to S$ $\Rightarrow$ A_S/S Azumaya
Will see later the converse is also true provided $R \to S$ is faithfully flat.

2.3. A/R, B/R azumaya $\Rightarrow$ $A \underset{R}{\otimes} B$ Azumaya

2.4. A/R, B/S Azumaya $\Rightarrow$ $A \otimes B / R \otimes S$ ($\otimes = \otimes_k$)
Also Azumaya (can be easily deduced from 2.2 + 2.3)

__Def__ (a) $A \sim B$ if there exist P, Q f.g projective generators over R: $A \otimes \mathrm{End}(P) \approx B \otimes \mathrm{End}(Q)$

Equivalently,

(b) $A \sim B$ if $A \otimes B^{op} \approx \mathrm{End}(P)$

__Pf__ of equivalence (b) $\to$ (a) $A \otimes B \otimes B^{op} = B \otimes \mathrm{End}(P) = A \otimes \mathrm{End}(B)$

(a) $\to$ (b): (a) implies that A and B are morita-equivalent.

Then $A = \mathrm{End}_B(P)$, $A \otimes B^{op} = \mathrm{End}_B(P) \otimes B^{op} = \mathrm{End}_{B \otimes B^{op}}(P \otimes B^{op})$

since $B \otimes B^{op} \sim R$, the latter can be replaced with $\mathrm{End}_R(Q)$.

__Def__ $Br(R)$ = equivalent classes of Azumaya algebras wrt $\otimes$.

$\alpha : R \to S$ induces $\alpha_* : Br(R) \to Br(S)$, $\alpha_*(A) = A_S$.

3. Splitting.

A/R Azumaya, $R \to S$ comm. ring homomorphism.

S splits A if A_S is trivial : $A \in \ker(Br(R) \to Br(S))$

Important property – Azumaya algebras can be split by especially nice extensions (étale).

3.1 Recall the case $R = k$.

Claim Let A/k be Azumaya, $\dim A = n^2$.
there exists ~~a maximal commutative subalgebra~~ an element $x \in A$ such that the subalgebra $k[x] \approx k[t]/(f)$ of A satisfies the following properties:

(1) $\deg f = n$

(2) $(f, f') = 1$ (no multiple roots – separability)

(3) $k[x]$ splits A

3.2 Universal splitting

Choose $R \to R_1$ s.t. A_{R_1} is a free R_1-module. $R_1 = \prod R_f$.
Define a functor assigning to $R_1 \to S$ the set

$$\{\theta : M_n(S) \xrightarrow{\sim} A_S\}$$

The functor can be easily representable: choose a base $(e_1, \dots, e_{n^2})$ of R_1-module A_{R_1}

Any θ is given by a collection of coefficients

$$\theta(e_{ij}) = \sum_{k=1}^{n^2} X_{ij}^k e_k ,$$

subject to lots of relations. Our representing algebra is generated by X_{ij}^k, modulo this relation.

We denote the representing algebra U^E and remember that

$$\mathrm{Hom}\,(U^E, S) = \{\theta : M_n(S) \to A_S\}$$

[we assume in this notation $R = R_1$; this is not very important]

~~Claim 4~~

<u>Def</u> Let R be noetherian, U a f.g. commutative R-algebra.

- U is called <u>smooth</u> if any diagram

$$\begin{tikzcd} & S \arrow[d] \\ U \arrow[ur, dotted] \arrow[r] & S/I \end{tikzcd}$$

of R-algebras, with $I^2 = 0$, lifts to a commutative triangle.

- U is called <u>étale</u> if this lifting is unique.

<u>Claim 1</u> U^E is smooth. This means, if A/S is Azumaya and $A_{S/I}$ is matrix algebra, then A is matrix algebra.

<u>Claim 2</u> Any smooth algebra is flat.
This is a nontrivial fact of commutative algebra.

<u>Claim 3</u> U^E is faithfully flat.
This is immediate as for any <u>prime ideal</u> $p \subset R$ there is a homomorphism $U^E \to k(p)$ where $k(p) = R_p / pR_p$.

Examples of étale algebras

① $R \longrightarrow R_f$ is étale:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ R_f \arrow[r] & S/I \end{tikzcd}$$

if $f \in R$ is invertible in S/I, it is invertible in S.

② Composition of étale morphisms is étale.

③ $R \longrightarrow R[x]/(f)$

where f is a polynomial with leading coeff. 1, satisfyng $(f, f') = 1$.

More generally, $S := R[x]/(f)$, the localization S_g is etale if f' is invertible in S_g (eg, $g = f'$)

<u>Proof</u>:

$$\begin{tikzcd} R \arrow[rr] \arrow[d] & & T \arrow[d, "p"] \\ S_g \arrow[rru, dashed] \arrow[rr, "\alpha"'] & & T/I \end{tikzcd}$$

one needs to find $t \in T$: $f(t) = 0$, $p(t) = \bar{t} := \alpha(X)$.

(The image of g will be automatically invertible)

Choose t_0: $p(t_0) = \bar{t}$, look for $t = t_0 + i$

$f(t) = f(t_0) + i \cdot f'(t_0)$ by Taylor formula ($i^2 = 0$).

Since $f(t_0) \in I$ and $f'(t_0)$ is invertible, there is a unique solution.

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We will prove later on the following

<u>Thm</u> Let A/R be Azumaya. For each $p \subset R$ prime there exists $f \notin p$ and $R_f \to S$ étale & finite as above splitting A.

Needs some preparation (and a lot of independently interesting stuff).

<u>4. Automorphisms</u> Recall Skolem-Noether:

Any automorphism of A/k Azumaya (over a field) is inner.

<u>Thm</u> Let R be such that $\mathrm{Pic}(R) = 0$ (eg, local). Then any autom. of A/R is inner.

<u>Pf</u> Let $\alpha: A \to A$ auto. Denote A_α the following A-A bimodule: $a x^* b = (a x \alpha(b))^*$ [here for $a \in A$ a^* denotes the element of A_α].

Since A is Azumaya, one has an equivalence $\mathrm{mod}(R) \rightleftarrows \mathrm{mod}(A^e)$ which yields

$$A_\alpha \approx A \otimes_R L \quad \text{where} \quad L = \mathrm{Hom}_{A^e}(A, A_\alpha)$$

This is an invertible R-module $\Rightarrow A_\alpha \approx A$ as bimodules.

Let $\varphi: A_\alpha \to A$ be an isomorphism. It has form $\varphi(a^*) = au$ where $u = \varphi(1^*)$. Then

$$\varphi(a^* b) = \varphi(a\alpha(b))^* = aub = a\alpha(b)u \Rightarrow \alpha(b) = ubu^{-1}$$

(invertible since φ is iso).

5. Characteristic polynomial

Let A/R Azumaya, $x \in A$. We wish to define $p_x \in R[t]$, the characteristic polynomial of x, generalizing the case $A = M_n(R)$.

Few remarks: 1 Assume $A \approx M_n(k)$. To define p_x, $x \in A$, one has to choose an isomorphism $A \xrightarrow{\sim} M_n(k)$. If one chooses another isomorphism, the difference gives an automorphism of $M_n(k)$ which is inner by Skolem-Noether. Since $p_x = p_{uxu^{-1}}$, we are done.

2 Assume A Azumaya over k, split over a Galois extension L, $G = Gal(L/k)$. For $x \in A$ one defines $p_x = p_{x \otimes 1} \in L[t]$ and one needs to check it is invariant to make sure $p_x \in k[t]$. Similar reasoning will be used in general, with replacement of G-action with <u>faithfully flat descent</u> (see below).

Choose a faithfully flat $\alpha: R \to S$ s.t. A_S is isomorphic to a matrix algebra over S. Define

$$p_x^S(t) = \det(t - x \otimes 1_S) \in S[t]$$

Let us check that the above definition does not depend on the isomorphism $\varphi: A_S \to M_n(S)$. If φ, ψ two such isomorphisms, $\alpha = \varphi\psi^{-1}$ is an automorphism of $M_n(S)$. For any localization $S \to S_q$ the image of α is inner by Skolem-Noether, so the images of two characteristic polynomials of x in each localization S_q of S coincide. This implies they coincide as elements of $S[t]$.

it remains to prove p_x^S belongs to $R[t]$ (and does not depend on S).

For any commutative triangle
$$\begin{tikzcd} & S \arrow[d, "\varphi"] \\ R \arrow[ur] \arrow[r] & T \end{tikzcd}$$
one has obviously $p_x^T = \varphi(p_x^S)$. Choose, for instance, $T = S \otimes_R S$ and $\varphi_0, \varphi_1 : S \to T$ given by the formulas $\varphi_0(s) = 1 \otimes s$, $\varphi_1(s) = s \otimes 1$.
One deduces
$$p_x^T = \varphi_0(p_x^S) = \varphi_1(p_x^S).$$

The rest follows from the following simple form of ff descent which will be discussed later.

<u>Prop</u> Let $R \to S$ be faithfully flat,

$$R \longrightarrow S \overset{\varphi_0}{\underset{\varphi_1}{\rightrightarrows}} S \otimes S$$

the diagram defined above. Then

$$R = \{ x \in S \mid \varphi_0(x) = \varphi_1(x) \}.$$

6. Splitting (continuation)

We are ready to prove the theorem formulated in 3.

Thm A/R Azumaya $^{[A:R]=n^2}$, $p \subset R$ prime $\Rightarrow \exists g \in R \sim p$ such that A is split over a finite étale subalgebra of A_g generated by one element over R_g:

$$S = R_g[x]/(f) \quad \text{, with } f' \text{ invertible.}$$

Proof Let $k = k(p)$. By the theory of Azumaya algebras over field, there exist $\bar{x} \in A_{k(p)}$ generating a splitting separable subalgebra

$$k(p)[x]/(f) \quad (\text{with invertible } f')$$

Lift $\bar{x}$ to an element $x \in A_{R_p}$ (R_p local ring with residue field k)

And consider the k-algebra generated by x. Denote it D. Its characteristic polynomial has degree n, so $x^n \in \mathrm{Span}\{1, x, \dots, x^{n-1}\}$. By Nakayama lemma, the set $\{1, x, \dots, x^{n-1}\}$ form a free basis over R_p and a subalgebra. It is separable since its reduction modulo the maximal ideal is separable.

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7. Faithfully flat descent

Let $f : R \longrightarrow S$ be faithfully flat

Def 7.1 The category $\mathrm{Sieve}(f)$ is the full subcategory of R-algebras T admitting a decomposition

$$\begin{tikzcd} & S \arrow[d] \\ R \arrow[ur, "f"] \arrow[dr] & \\ & T \end{tikzcd}$$

Def. An R-module given S-locally is the following assignment:

1) for each $T \in \mathrm{Sieve}(f)$ a T-module M_T
2) for each $\alpha : T \to T'$ in $\mathrm{Sieve}(f)$ ~~an isomorphism~~ $\cancel{\alpha_* M}$ a map $M_T \longrightarrow M_{T'}$ of T-modules inducing an isomorphism of T'-modules $T' \otimes_T M_T \to M_{T'}$.
3) such that for $T \xrightarrow{\alpha} T' \xrightarrow{\alpha'} T''$ the diagram

$$\begin{tikzcd} M_T \arrow[rr] \arrow[dr] & & M_{T'} \arrow[dl] \\ & M_{T''} & \end{tikzcd}$$

is commutative.

Example Any R-module M defines an R-module given S-locally by the assignment

$$T \longmapsto M_T = T \otimes_R M$$

~~R~~ R-modules given S-locally form a category denoted $\mathrm{Mod}_R^{S\text{-loc}}$ (local notation).

One has a functor $F : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R^{S\text{-loc}}$ described above.

Define a functor $G : \mathrm{Mod}_R^{S\text{-loc}} \longrightarrow \mathrm{Mod}_R$ as follows

Let $M = \{M_T\} \in \mathrm{Mod}_R^{S\text{-loc}}$. We define

$$G(M) = \mathrm{Ker}\left(M_S \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} M_{S \otimes_R S} \right) \quad \text{where}$$

$\partial^0, \partial^1 : S \to S \otimes_R S$ are two maps in Sieve(f) defined as $\partial^0(s) = 1 \otimes s$, $\partial^1(s) = s \otimes 1$.

Note that we have not yet used the fact that f is f.flat.

<u>Lemma</u> The functor G is right adjoint to F

<u>Proof</u> Let $N \in \mathrm{Mod}_R$ and $M \in \mathrm{Mod}_R^{S\text{-loc}}$. One has

$\varphi \in \mathrm{Hom}(F(N), M)$ is given by a compatible collection of maps $T \otimes_R N \to M_T$ or, what is the same, of maps $N \to M_T$ of R-modules. compatibility means that for any $\alpha : T \to T'$ the diagram

$$\begin{tikzcd} & M_T \arrow[d] \\ N \arrow[ur] \arrow[r] & M_{T'} \end{tikzcd}$$

commutes.

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This huge collection of data can be simplified.
It contains a map $N \longrightarrow M_S$; any $T \in \mathrm{Sieve}(f)$ admits a decomposition $R \to S \to T$, so the ~~map~~ map $N \to M_T$ is defined as composition

$$N \longrightarrow M_S \longrightarrow M_T$$

However, the map $N \to M_S$ should satisfy some extra requirement: if the compositions $R \to S \rightrightarrows T$ coïncide, the compositions $N \to M_S \rightrightarrows M_T$ should coïncide as well.

A pair of maps $S \rightrightarrows T$ amounts to a single map $S \otimes_R S \to T$, so the only condition is that

$$N \to M_S \rightrightarrows M_{S\otimes S} \quad \text{coincide,}$$

that is the image of N in M_S lies in $G(M)$. $\square$

see p. 12bis

Thm (ff descent)
Assume $f: R \to S$ faithfully flat. Then (F, G) is an equivalence of categories.

Before the proof – corollaries

Cor 1 Let $N \in \mathrm{Mod}_R$. Then $N = \{x \in N_S \mid x \otimes 1 = 1 \otimes x\}$.
In effect, this is the claim $GF(N) = N$.

Cor 2 Let $N, N' \in \mathrm{Mod}_R$, $\alpha: N_S \to N'_S$ satisfies $\alpha \otimes 1 = 1 \otimes \alpha : N_{S\otimes S} \to N'_{S\otimes S}$. Then α is induced by a map $N \to N'$.

FF descent is a very important theorem having uncountable number of applications.

Proof 1. Case 1. $f: R \to S$ splits, i.e. $\exists\, g: S \to R$ s.t. $g \circ f = 1_R$. Then the theorem is obvious as $\mathrm{Sieve}(f)$ is the whole category of R-algebras having the initial object.

2. General case. Here it is convenient to use the description of $\mathrm{Mod}_R^{S\text{-loc}}$ via descent data.

The base-change functor $N \mapsto S \otimes_R N \;: \mathrm{Mod}_R \to \mathrm{Mod}_S$ defines a similar base-change functor $\mathrm{Mod}_R^{S\text{-loc}} \to \mathrm{Mod}_S^{S\otimes S\text{-loc}}$.

Since S is faithfully flat, the base change functor is fully faithful. The adjoint pair

$$\mathrm{Mod}_S \rightleftarrows \mathrm{Mod}_S^{S\otimes S\text{-loc}}$$

is equivalence since $S \to S\otimes S$ splits.
Therefore, the connected homomorphisms $FG(M) \to M$ and $N \to GF(N)$ are isomorphisms.

$\boxtimes$

Cor Let $R \to S$ be faithfully flat. Then the functor from R-algebras to S-algebras endowed with descent data, is an equivalence.

$\boxtimes$

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The category $\mathrm{Mod}_R^{S\text{-loc}}$ can be replaced with a more 'tractable' one.

Let $M = \{M_T\} \in \mathrm{Mod}_R^{S\text{-loc}}$.

Put $N := M_S$. The maps $\partial_0, \partial_1 : S \to S \otimes_R S$ identify $M_{S\otimes S}$ with $S \otimes N$ and with $N \otimes S$. This gives a canonical isomorphism $\theta : S \otimes N \xrightarrow{\sim} N \otimes S$.

Furthermore, base change along three maps $\partial_0, \partial_1, \partial_2 : S\otimes S \to S\otimes S \otimes S$ defined by $x \otimes y \mapsto 1 \otimes x \otimes y,\ x \otimes 1 \otimes y,\ x \otimes y \otimes 1$ respectively, carries θ into isomorphisms $\theta_0, \theta_1, \theta_2$ connected by the commutative diagram

$$\begin{tikzcd} S\otimes S\otimes N \arrow[rr, "\theta_0"] \arrow[dr, "\theta_1"'] & & S\otimes N\otimes S \arrow[dl, "\theta_2"] \\ & N\otimes S\otimes S & \end{tikzcd}$$

The map $\theta : S\otimes N \xrightarrow{\sim} N \otimes S$ of $S\otimes S$-modules satisfying the condition $\theta_1 = \theta_2 \circ \theta_0$ (cocycle condition), is called <u>descent datum</u>. We see that each $M \in \mathrm{Mod}_R^{S\text{-loc}}$ determines ~~a descent datum~~ an S-module $N := M_S$ with a descent datum. We claim that $\mathrm{Mod}_R^{S\text{-loc}}$ is equivalent to the category of S-modules with descent datum.

Let us indicate how to reconstruct $\{M_T\}$ from (N, θ). Choose for each $T \in \mathrm{Sieve}(R)$ a decomposition

$$R \to S \xrightarrow{g_T} T$$

define $M_T = T \otimes_S N$ (with respect to g_T).

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For each $\alpha : T \to T'$ we have a diagram

$$\begin{tikzcd} & T \arrow[d, "\alpha"] \\ S \arrow[ur, "g_T"] \arrow[r, "g_{T'}"'] & T' \end{tikzcd}$$

– of course, not commutative

The pair of maps $S \to T'$ gives rise to $g : S \otimes S \to T'$ so that $g \circ \partial^0 = \alpha \circ g_T$, $g \circ \partial^1 = g_{T'}$.

This allows one to define $T' \otimes_T M_T = T' \otimes_T T \otimes_S N \to T' \otimes_S N = M_{T'}$ as $T' \otimes_{S \otimes S} \theta$.

~~One~~ To complete the construction, one has to check that the maps $T' \otimes_T M_T \to M_{T'}$ behave well with respect to the compositions. This will result from the cocycle condition.

8. Amitsur cohomology

Amitsur cohomology is an algebraic version of Čech cohomology. Let R be a commutative ring, S a faithfully flat R-algebra. S defines 'an open covering' of the 'space defined by R.' ~~$S\otimes$~~ Here is the table of correspondences which can be made precise

Topology	Algebra
$X = \cup U_i$	$R \longrightarrow S$
$\coprod U_i$	S
$\coprod U_i \cap U_j$	$S \otimes_R S$
$\coprod U_{i_1} \cap \dots \cap U_{i_K}$	$S^{\otimes K}$
(Pre) Sheaf $\mathcal{F}$ on X	Functor F on Alg_R
$\prod \mathcal{F}(U_i)$	$F(S)$
$\prod \mathcal{F}(U_{i_1} \cap \dots \cap U_{i_K})$	$F(S^{\otimes k})$
$\check{H}^*(\mathcal{U}, \mathcal{F})$	$H^*(S/R, F)$

Here are the appropriate definitions

Let $R \to S$ be faithfully flat, $F : R\text{-alg} \longrightarrow Ab$ a functor to abelian groups

<u>Examples</u>: ~~$F(T)$~~ $F(T) = (T, +)$ additive group of T

$F(T) = (T^*, \cdot)$ invertible elements of T

Standard names for these functors: G_a, G_m.

Amitsur complex of S/R with coefficient in F is defined as follows

$$C^n(S/R, F) = F(S^{\otimes n+1})$$

$d: C^n(S/R, F) \to C^{n+1}(S/R, F)$ is defined as

$$d = \sum_{k=0}^{n} (-1)^k F(\partial^k), \quad \partial^k : S^{\otimes n+1} \to S^{\otimes n+2} \text{ is}$$

defined by $\partial^k(s_0 \otimes \cdots \otimes s_n) = s_0 \otimes \cdots \otimes \underset{\overset{\uparrow}{k}}{1} \otimes \cdots \otimes s_n$

Amitsur cohomology is defined as

$$H^n(S/R, F) = H^n(C(S/R, F))$$

<u>Lower degrees</u> If F is just a functor to Sets, one still can define $H^0(S/R, F)$ by the formula

$$H^0(S/R, F) = \mathrm{Ker}\left(F(S) \overset{F(\partial^0)}{\underset{F(\partial^1)}{\rightrightarrows}} F(S \otimes S) \right)$$

In this case H^0 is a set.

<u>Degree 1</u> If F is a functor to groups (not necessarily abelian groups), one still can define $H^1(S/R, F)$ which is a pointed set. It is defined as a quotient of a set $Z^1(S/R, F)$ by an equivalence relation (defined below).

An element $\theta \in F(S \otimes S)$ belongs to Z^1 iff

$$\partial^1(\theta) = \partial^2(\theta) \cdot \partial^0(\theta)$$

(looks familiar?)

Two elements $\theta, \theta' \in Z^1$ are equivalent if there exists $u \in F(S)$ so that

$$\theta' = \partial^1(u^{-1})\, \theta\, \partial^0(u) .$$

<u>Example</u> $F(T) = GL(n, T)$. The set $H^1(S/R, GL(n))$ is the set of isomorphism classes of projective R-modules of rank n, splitting over S.

In particular, $H^1(S/R, G_m) = \mathrm{Ker}(\mathrm{Pic}(R) \to \mathrm{Pic}(S))$

<u>Example</u> If $\mathrm{Pic}(S) = 0$, $H^1(S/R, PGL(n))$ is the set of isomorphism classes of Azumaya algebras $/R$ such that $A_S \approx M_n(S)$ ~~(that is, split over S~~

Here $PGL(n)(S)$ is the group of inner automorphisms of $M_n(S)$.

Finally, similarly to what is done with the conventional Čech cohomology, it makes sense to pass to finer covering. One defines

Better notation: $H^{\cdot}(\mathrm{Spec}\, R_{\acute{e}t}, F)$

$$H^{\cdot}_{et}(R, F) = \underset{\substack{S/R \text{ étale} \\ \text{faithful}}}{\mathrm{colim}}\, H^{\cdot}(S/R, F)$$

$$H^{\cdot}_{flat}(R, F) = \underset{S/R \text{ f.flat}}{\mathrm{colim}}\, H^{\cdot}(S/R, F)$$

To be on the safe side, let us recall the notion of colimit:

For each $S \in \{\text{faithfully flat } R\text{-algebras}\}$ one has a set $H(S) := \check{H}(S/R, F)$

Then

$$\operatorname{colim} H = \coprod_S H(S) / \sim$$

where the equivalence relation is generated by the identification of $a \in H(S)$ with $\alpha_*(a) \in H(S')$ for any $\alpha : S \to S'$ (and $\alpha_* : H(S) \to H(S')$)

Lemma

$\boxed{\text{Calculation}}$ $H^1(R, PGL(n))$ is the set of isomorphism classes of Azumaya algebras of rank n^2.

Let $\alpha \in H^1(S/R, PGL(n))$. Its representative is an inner automorphism of $M_n(S \otimes S)$ satisfying the cocycle condition. This defines descent datum for S-algebra $M_n(S)$, and, by descent, an algebra over R. It is obviously Azumaya of rank n^2.

In the opposite direction, let A be Azumaya of rank n^2. There exists an étale covering $R \to S$ such that $A_S \simeq M_n(S)$. Descent datum for $M_n(S)$ defining A_S, is an automorphism of $M_n(S \otimes S)$ - not necessarily inner. However, such an automorphism determines an invertible $S \otimes S$-module $\mathcal{L}(\alpha)$. Choose an étale map $S \otimes S \to T$ trivializing $\mathcal{L}(\alpha)$, and we have found an etale covering $R \to T$ for which the descent datum is inner.

Some abstract nonsense (sheaf theory?)

Let $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ be a sequence of three functors and homomorphisms between them (we assume them abelian). The sequence is <u>exact</u> if

1) $\beta \circ \alpha = 0$
2) $\forall\, S/R \quad \forall\, \gamma \in G(S) : \beta(\gamma) = 0 \quad \exists\; S \xrightarrow{u} T \;\; \exists \varphi \in F(T)$ such that $\alpha(\varphi) = u_*(\gamma)$

Remark The second condition replaces the expected "$\forall \gamma\; \exists \varphi : \alpha(\varphi) = \gamma$" in the way this is done in sheaf theory.

Thm (formal) A short exact sequence

$$0 \to F \to G \to H \to 0$$

gives rise to a long exact sequence of cohomology

$$0 \to H^0(R,F) \to H^0(R,G) \to H^0(R,H) \to H^1(R,F) \to H^1(R,G) \to \dots$$

A variant of this makes sense when some non-commutativity is allowed (see below).

Basic short exact sequence

Lemma $1 \to G_m \to GL(n) \to PGL(n) \to 1$
is exact, (Obvious)

Cor One has an exact sequence

$$1 \to G_m(R) \to GL_n(R) \to PGL_n(R) \to H^1(R, G_m) \to H^1(R, GL_n)$$

$$\to H^1(R, PGL_n) \xrightarrow{c} H^2(R, G_m).$$ [cannot go on as GL_n is not commutative]

Let us interpret these cohomology groups.
$H^1(R, PGL_n)$ is the set of isomorphism classes of Azumaya algebras of rank n^2.
$H^1(R, G_m) = Pic(R)$
$H^1(R, GL_n)$ is the set of isomorphism classes of ~~ran~~ projective R-modules of rank n.

The map $H^1(R, GL_n) \to H^1(R, PGL_n)$ assigns to P the class of $End(P)$.

Cor Let $[A] \in H^1(R, PGL_n)$. Then $c(A) = 1$ iff A is split.

<u>Lemma</u> $c(A\otimes B) = c(A)\cdot c(B)$ $\boxtimes$

<u>Corollary</u> The above short exact sequence defines an injective map
$$c: Br(R) \to H^2(R, G_m).$$

<u>Thm (Gabber, 1980)</u> The image of c is $H^2(R, G_m)_{tors}$.

Let us check $Br(R)$ is a torsion group.

<u>Lemma</u> $1 \to \mu_n \to SL(n) \to PGL(n) \to 1$ is exact, where $\mu_n(R) = \{x\in R \mid x^n = 1\}$

In flat topology!!!

<u>Proof</u> Important: surjectivity. Let S/R be étale with $\bar{A}\in PGL_n(S)$. It is in general impossible to lift $\bar{A}$ to $A\in SL_n(S)$. Let us lift $\bar{A}$ to $A\in GL_n(S)$ and try to correct the lifting. Let $x = \det(A)\in S$. We have to find a faithfully flat extension $S\to T$ such that $x\in T$ has n-th root. That is easy: $T = S[t]/(t^n - x)$

<u>Note:</u> one cannot expect to find such T to be étale. Thus, the sequence is exact in flat topology, but is not exact in étale topology.

<u>Cor</u> Similarly to $c: Br(R)\to H^2_{et}(R, G_m)$ one can define the map in flat topology which is also injective and factors through $H^2_{fl}(R, \mu_n)$. Thus, any Azumaya algebra of $rk = n^2$ is annihilated by n in the Brauer group.