The #12equations challenge

day 1.

$$Tr(A_1 \otimes \cdots \otimes A_n \circ \sigma) = Tr(A_1 \cdots A_n),$$

where $A_1, \ldots, A_n$ are linear operators on a finite dimensional vector space V and σ is the cyclic permutation acting on $V^{\otimes n}$, $\sigma : v_1 \otimes \cdots v_n \mapsto v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$.

day 2.

$$\chi_1(ab) = \chi_1(a)\chi_1(b),$$

$$\chi_2(a)\chi_2(b)\chi_2(c) = \chi_2(ab)\chi_2(c) + \chi_2(bc)\chi_2(a) + \chi_2(ac)\chi_2(b) - \chi_2(abc) - \chi_2(acb).$$

the identities satisfied by characters of a group (or associative algebra) representations of degree 1 and 2. A similar equation can be written for a degree d character for any d, the proof following immediately from the identity posted yesterday.

day 3. Not all good things in life are about matrix multiplication (or so they say) but still, I am not leaving it alone just yet.

Amitsur-Levitzki Theorem. The second equivalent form involves vanishing of cocycles c_k for large k, the ones that don't vanish freely generate the cohomology algebra of $gl(n)$. So in away the AL identity reflects topology of the Lie group of unitary matrices. Is there a way to make this connection closer?

$$\sum_{\sigma \in S_{2n}} sgn(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(2n)} = 0,$$

$$A_i \in Mat_n.$$

Set

$$c_k(A_1, \ldots, A_{2k-1}) = Tr(\sum_{\sigma \in S_{2k-1}} sgn(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(2k+1)}),$$

then $c_k = 0$ for $k \geq n$, $A_i \in Mat_n$.

day 4, the Drinfeld curve.

$$X = \{((x, y) \mid xy^q - x^q y = 1\},$$

$$|Aut(X)| \geq q(q^2 - 1)(q + 1)/2 > 42(2g - 2), \qquad g = q(q - 1)/2.$$

A remarkable curve over (an algebraically closed) field of positive characteristic p, cut out by the displayed equation where q is a power of p. Its automorphism group contains $SL(2, q)$ and the group of norm one elements in the quadratic extension of F_q, it is way bigger then the group of automorphisms of any curve of the same genus $g = q(q-1)/2$ over complex numbers: the latter is bounded by $42(2g-2)$ according to the classical Hurwitz bound (we assume q is not too small). X can be used to realize geometrically representations of SL(2,q), its generalization known as Deligne-Lusztig varieties do the same for other finite Chevalley groups.

day 5.

Plato thought regular icosahedra to be what water is made of, in Kepler's cosmography a regular icosahedron supported the sphere on which the Earth rotated, though its role in today's post is rather related to ideas that came later. If Γ' is its group of symmetries, and Γ is the preimage of Γ' in $SU(2)$, then the quotient of the complex plane by Γ is the simple singularity of type E8, so the second cohomology of its minimal resolution realizes the E8 lattice. That singular variety X sits in the "most" exceptional simple Lie algebra E8 as the transversal slice in the set of nilpotent elements to the codimension 2 nilpotent orbit. Is there a natural construction producing the E8 Lie algebra from X (so ultimately from the icosahedron)?

$$X = \{(x, y, z) \mid x^2 + y^3 + z^5 = 0\} \cong \mathbb{C}^2/\Gamma,$$

$$\widetilde{X} \to X \text{ is the minimal resolution,}$$

$$H^2(\widetilde{X}) \cong \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 \mid \sum x_i = 0 \mod 2\}.$$

day 6.

There are many wonders involving Fourier transform over a finite field, this post is about a (somewhat randomly selected) couple of examples.

The first formula is about cardinality of the set of rational points in the intersection of a smooth projective variety over a finite field of q elements with a hyperplane not contained in the projectively dual variety. The answer is a constant C which is actually the trace of Frobenius on the primitive part of cohomology of X plus the "varying term" which is the trace of Frobenius in the new cohomology of the hyperplane section, since by Deligne all eigenvalues of Frobenius in that space have absolute value $q^{d/2}$, the term is bounded by a constant (dimension of the new cohomology) times $q^{d/2}$.

The second formula is about Fourier transform of the characteristic function of the set of nilpotent matrices, the answer is very explicit: it vanishes on nonsemisimple elements and equals an explicit power of q up to an explicit sign (not made explicit here). The formula generalizes to Lie algebras of reductive groups, the group analogue of that function is the Steinberg character.

day 7. An idiosyncratic proof of the Riemann-Roch Theorem for flag varieties a.k.a. Weyl dimension formula.

In this special case it can be rewritten in the form of the last displayed equality, multiplication by the Todd class has turned into twisting with the square root of the anti-canonical class, i.e. tensoring with $\mathcal{O}(\rho)$. The base field has characteristic p and the key fact is that Frobenius pushforward of the line bundle of half forms $O(-\rho)$ is the sum of p^d copies of the same bundle (d=dim(G/B)). This implies that the functional "Euler characteristic of the twist by half forms" is an eigenvector with respect to Frobenius pull back, so it has to be proportional to the integral of Chern character.

$$\begin{gathered} Fr_*(\mathcal{O}_{G/B}(-\rho)) \cong \mathcal{O}_{G/B}(-\rho)^{\oplus p^d} \Rightarrow \\ \chi(Fr^*(\mathcal{F}) \otimes \mathcal{O}(-\rho)) = p^d \chi(\mathcal{F} \otimes \mathcal{O}(-\rho)) \Rightarrow \\ \chi(\mathcal{F} \otimes \mathcal{O}(-\rho)) = \text{const} \int ch(\mathcal{F}) \Rightarrow \\ \chi(\mathcal{F}) \int ch(\mathcal{O}(\rho)) = \int ch(\mathcal{F} \otimes \mathcal{O}(\rho)). \end{gathered}$$

day 8. Snaith's (?) geometric proof of Brauer induction theorem.

Just a pretty argument I came across when leafing through a random book in the Givat Ram library some time in graduate school that made an impression. The library used to be in the ground floor with those narrow windows that are still there today, that year a little bird made a nest and laid an egg in one of the windows :)

Let us prove a weak form of BIT saying any representation of a finite group G is an integral combination of characters induced from one dimensional representations. The logic is weird: if there happens to exist a single irreducible representation that's not induced from a character then we deduce that the trivial representation is a combination of reps induced from proper subgroups, so everything follows by induction. To show that implication one considers the action of G on the space X parametrizing splitting of the representation space V as an orthogonal (with respect to an invariant Hermitian form) sum of unordered lines. Rational homology of X is one dimensional, and if V is not induced then G acts on X without fixed points. Now an equivariant complex computing cohomology of X is a resolution of the trivial representation by nontrivial permutation modules, i.e. modules induced from smaller subgroups!

$$X^G = \emptyset \Rightarrow \exists : \quad 0 \to C^0 \to C^1 \to \cdots C^n \to 0,$$
$$H^*(C^\bullet) = H^*(X) \ \& \ C^i = \oplus Ind_{H_{ij}}^G(?), \ H_{ij} \subsetneq G;$$

$$X = U(n)/N(T), \quad H^i(X, \mathbb{Q}) = 0 \ \text{ for } i > 0;$$
$$\text{for } \rho : G \to U(n) \text{ irreducible}, \quad X^G = \emptyset \text{ or } \rho \cong Ind_H^G(\chi).$$

day 9. Short proof of Adams-Riemann-Roch in characteristic p after Pink and Rossler.

Adams-Riemann-Roch tells how Adams operations commutes with proper direct image, just as Grothendieck-Riemann-Roch does the same for Chern character (and the former formally implies the latter).

Now, in characteristic p the p-th Adams operation is nothing but pull back under Frobenius, while Adams genus (parallel to the Todd genus in the GRR story) is the class of $Fr^*Fr_*(O)$! The last point is quite transparent: Adams genus is associated to the series (polynomial) $1 + t + ... + t^{p-1}$ and the fiber of $Fr^*Fr_*(O)$ at a point x is functions on Frobenius neighborhood of x. With these two observations in hand, projection formula and staring at the Cartesian square formed from the $f' : X' \to Y'$ and $Fr : Y \to Y'$ (where $f : X \to Y$ is a proper morphism of smooth varieties over a characteristic p field, and f', X', Y' are Frobenius twists of f, X, Y) gives a short and easy proof.

$$A_p(X) = Fr^*Fr_*(\mathcal{O}_X) \Rightarrow$$

$$[Fr^*(Rf_*(\mathcal{F})) \otimes A_p(Y)] = [Rf_* \left(Fr^*(\mathcal{F}) \otimes A_p(X)\right)]]$$

day 10, Cartier isomorphism.

For a smooth variety X over a characteristic p field cohomology sheaves of the De Rham complex are identified with forms on the Frobenius twist of X.

There are many related facts and generalizations. The best known is the famous algebraic proof of Hodge theorem by Deligne - Illusie, but there are also more recent ones: twisted version by Ogus and Vologdsky, noncommutative generalization by Kaledin... Apparently, the proper context for it is the mod p and p-adic Hodge theory, I wish I understood it well enough to express by a formula (or twelve formulas)!

$$H^i(\Omega^\bullet_X, d) \cong \Omega^i_{X^{(1)}},$$

$$[f^p \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_i}{z_i}] \mapsto f \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_i}{z_i}.$$

day 11, Frobenius splitting.

An algebraic variety X over a field of characteristic p is Frobenius split if the structural map from O to Fr_*O admits a left inverse. Then cohomology of a line bundle L is a direct summand in cohomology of L^{p^n}, in particular if X is projective and L is ample then higher cohomology vanishes. There are also splittings compatible with a closed subvariety whose existence implies cohomology vanishing involving the ideal of that subvariety.

I have always felt there is something magical about it, the cohomology vanishing applications should not be its only use. Any ideas?

$$i : \mathcal{O}_X \to Fr_*\mathcal{O}_X;\ \exists j : Fr_*\mathcal{O}_X \to \mathcal{O}_X,\ j \circ i = Id \Rightarrow H^i(X, L) \subset H^i(X, L^{\otimes p^n})$$

$$\Rightarrow H^{>0}(X, L) = 0 \text{ if } X \text{ is projective, } L \text{ is ample.}$$

day 12, Grothendieck-Katz p-curvature conjecture.

In one of these postings I wrote about a little bird and where it's nested in 1995 or so, but this one is no small bird.

Starting with a vector bundle with a flat connection on a smooth variety defined, say, over a number field, one can reduce it modulo almost any prime and then compute the p-curvature. If one gets zero almost always then Grothendieck and Katz predict that the connection trivializes on a finite cover.

If it's not zero, one gets a mysterious invariant, a collection of subvarieties in the cotangent bundle, one for (almost) every prime. It's known these are Lagrangian, but I don't think much more is known beyond that, despite of some intriguing ideas proposed by Kontsevich.

With that my journey is over, many thanks for "likes" and comments, and thanks to Anton Mellit for the opportunity!

$$\nabla^p_\xi = \nabla_{\xi^{[p]}} \forall \xi, \text{ almost all } p \overset{?}{\Rightarrow} \exists \text{ étale } f : Y \to X,\ f^*(\mathcal{E}, \nabla) \cong (\mathcal{O}^n, d)$$