Random Walk Models for Velocity-Shear Dispersion in Fluids

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Abstract

The topic of this project concerns the theory of shear dispersion and its connection to random walks. In shear dispersion processes, a random walk is coupled with a unidirectional current and the results are often intriguing and quite different from what would be expected if either of the two were considered independently of the other. The first part of the project paper introduces G.I. Taylor's asymptotic theory of shear dispersion in an closed channel. The second part looks at two cases of shear dispersion that give rise to anomalous diffusive scaling: power law shear flows and random velocity fields. The third part of the project is an analysis of the case of a sinusoidally varying velocity profile, $u(y)$, with a transverse random walk. A combination of dimensional analysis and numerical simulation is used to obtain the dependence $\sigma_x^2(t)$.

Chapter 1

Shear Dispersion: A Review of Some Interesting Features

1.1 Introduction to Taylor Dispersion and Connection to Random Walks

The subject of this project is the physical phenomenon that was first analyzed by the great 20th century physicist G.I. Taylor and bears his name: Taylor dispersion[6]. The process is the result of coupling between a unidirectional drift and a one-dimensional random walk. The drift is a steady shear flow, $u_x = u_x(y)$, with y the coordinate in the direction of the walk. The random walk occurs transverse to the drift, the position of the walker with time being described by the random variable $y = y(t)$. The longitudinal velocity experienced by each walker varies over its trajectory, and since the trajectory is a random one, the range of velocities experienced will vary between walkers. The result is that the longitudinal position becomes a random variable with variance, σ_x^2, that grows with time. The scaling of σ_x^2, however is not trivial to obtain and depends a great deal on the character of the drift, $u_x(y)$,as will be discussed.

1.2 Taylor's Analysis

Taylor's original analysis was focused on the spreading of a solute in laminar flow in a tube. The profile of the velocity in a tube of radius a is that of

Poisseuille flow, varying parabolically with radial distance from the center, $u(r) = u_o(1 - r^2/a^2)$. He argued that given a sufficient time, each particle performing a Brownian motion will have traversed the extent of the tube, randomly sampling all of the velocities. After such time, the mean displacement will be given by $\langle x \rangle = \langle u \rangle \cdot t$, where

$$\langle u \rangle = \frac{1}{2R} \int_{-R}^{R} u(r) dr$$

He then found, using analysis to be described below, that the variance, σ_x^2, in longitudinal position of the solute increased linearly with time. Thus, the longitudinal dispersion process exhibited normal diffusive scaling and Taylor was able to calculate the effective "dispersion" constant,

$$K = \frac{a^2 u_o^2}{192D}$$

with D the molecular diffusion constant of the particular solute.

Instead of presenting Taylor's exact analysis for the tube, the case of dispersion in a fluid flowing between two parallel plates will be discussed instead. The problem is the same in every aspect except that rectangular coordinates can be used rather than radial coordinates, making the diffusion equation take the more familiar form. The details of Taylor's analysis are well-presented by Fischer et. al. [3] and their general presentation will be followed below.

Taylor began with the diffusion (Fokker-Planck) equation,

$$\frac{\partial}{\partial t}\left(C' + \overline{C}\right) + \frac{\partial}{\partial x}\left(u' + \overline{u}\right) = D\left[\frac{\partial^2}{\partial x^2}\left(C' + \overline{C}\right) + \frac{\partial^2 C'}{\partial x^2}\right] \tag{1.1}$$

where C, the solute concentration, and u are both resolved into their respective mean ($\overline{C}$, $\overline{u}$) and spatially fluctuating (C' and u') components, with respect to transverse coordinate. He then transformed the equation to a reference frame moving with the mean velocity, $\xi = x - \overline{u}t$, so (1.1) becomes

$$\frac{\partial}{\partial \tau}\left(C' + \overline{C}\right) + u'\frac{\partial}{\partial \xi}\left(C' + \overline{C}\right) = D\left[\frac{\partial^2}{\partial \xi^2}\left(C' + \overline{C}\right) + \frac{\partial^2 C'}{\partial y^2}\right] \tag{1.2}$$

He neglected the longitudinal molecular diffusion term, $D\frac{\partial^2}{\partial \xi^2}\left(C' + \overline{C}\right)$,since he expected the variance due to longitudinal dispersion to be much more significant than that from molecular diffusion. He then made the assumption that the concentration profile was a slowly varying function of lateral position so that the fluctuations were small compared with the mean, $C' << \overline{C}$. It then followed that $\frac{\partial C'}{\partial \tau} << \frac{\partial \overline{C}}{\partial \tau}$ and $u'\frac{\partial C'}{\partial \xi} << u'\frac{\partial \overline{C}}{\partial \xi}$. His final simplification, perhaps seemingly the greatest leap, was to argue that $\frac{\partial \overline{C}}{\partial \tau}$ was small in magnitude compared with $u'\frac{\partial \overline{C}}{\partial \xi}$ and $D\frac{\partial^2 C'}{\partial y^2}$. This is unusual since in the dispersion process, the term $\frac{\partial \overline{C}}{\partial \tau}$ is precisely what is ultimately sought. However, Taylor argued that in the moving reference frame, after a long enough period of time, an equilibrium existed between the production of transverse concentration gradient due to the velocity gradient (shear) and the dissipation of those gradients by molecular diffusion, and he obtained the simple balance,

$$u'\frac{\partial \overline{C}}{\partial \xi} = D\frac{\partial^2 C'}{\partial y^2} \tag{1.3}$$

This was the foundation of Taylor's analysis, and he easily solved (1.3) which has the no-flux boundary conditions at the walls,

$$\frac{\partial C'}{\partial y} = 0 \qquad at \;\; y = 0, h$$

where h is the distance between the parallel plates. The solution, obtained by integrating twice on y, and then translating back into the original frame of reference, is

$$C'(y) = \frac{1}{D}\frac{\partial \overline{C}}{\partial x}\int_0^y \int_0^y u' dy dy + C'(0).$$

The total mass flux in the direction of flow can be found by integrating the instantaneous flux, $J(y) = u'C'$ over the total width, which gives

$$J_{total} = \int_0^h u'C'dy = \frac{1}{D}\frac{\partial \overline{C}}{\partial x}\int_0^h u' \int_0^y \int_0^y u' dy dy dy \tag{1.4}$$

$\left(\int_0^h u'C'(0)dy = 0 \;\; since \;\; \int_0^h u'dy = 0\right)$. Inspection of 1.4 shows that the solute flux is proportional to its longitudinal concentration gradient, $J \sim \frac{\partial C}{\partial x}$, which implies normal diffusive scaling. Taylor realized that it was thus appropriate to define an effective longitudinal dispersion constant, K, and since in terms of a Fickian diffusion process,

$$J_{total} = -hK\frac{\partial C}{\partial x},$$

it follows that

$$K = -\frac{1}{D}\int_0^h u' \int_0^y \int_0^y u' dy dy dy$$

The effective dispersion constant is thus inversely proportional to the molecular diffusion constant, $K_x \sim \frac{1}{D}$.

The reason for the inverse scaling with molecular diffusion can be seen by considering that for small D, solute particles tend to remain on the same velocity streamlines for much longer than for large D, and thus the rate of spread between particles on fast vs. slow moving streamlines is much greater than if the particles could readily exchange between streamlines. In this way, both the details of the random walk as well as the velocity field are coupled and directly impact σ_x in a way that lends a much greater complexity to the process than it might at first seem.

The major insight of Taylor's analysis is also the biggest caveat: his primary assumption of a balance between shear and molecular diffusivity. This is clearly only true in the asymptotic limit. For early times, before the random walkers have sampled the entire velocity distribution, the contribution of velocity to σ_x is dominant and longitudinal variance grows faster than diffusively,

$$\sigma_x \sim t^\nu \qquad \nu > 1/2$$

This simply means that the $x-$ displacements are correlated for small times, $t < \tau$, where τ is the integral of the normalized velocity autocorrelation function, often called the integral time scale,

$$\tau = \frac{\int_0^\infty \left\langle \langle u(0\rangle_y \langle u(t)\rangle_y \right\rangle dt}{\left\langle \langle u\rangle_y \right\rangle_t \left\langle \langle u\rangle_y \right\rangle_t} \tag{1.5}$$

($\langle\rangle_y$ and $\langle\rangle_t$ are space- and time- averages). Thus Taylor's theory of dispersion ties directly back to his theory of "Diffusion by Continuous Movements[7]" where he used the velocity correlation to characterize the diffusion constant in a turbulent fluid.

The pre-asymptotic behavior of Taylor dispersion can be understood by looking at results for regimes in which the velocity profile is unbounded in the transverse direction and the random walk never reaches a state where all velocities are representatively sampled. In these cases, discussed below, an asymptotic diffusive limit may never be reached and anomalous scaling, $\sigma_x \sim t^\nu$, $\nu > 1/2$, can result.

1.3 Dispersion in Unbounded Flows; Anomalous Diffusion

Unbounded shear flows, wherein the velocity profile grows in either transverse direction according to a power law, $|u(y)| \sim y^{\beta}$ show very interesting asymptotic scaling for the longitudinal variance of Brownian particles. Such flows were addressed in a paper by Ben-Naim et. al. [1] from the perspective of random walks, and their interesting results are discussed here.

The authors found anomalous scaling for σ_x for these regimes and gave the following scaling arguments to account for it. For a random walk starting at the origin, the transverse position scales as, $y(t) \sim \sqrt{D_y t}$. Since $|u(y)|$ is an increasing function of y, the longitudinal drift experienced by the walkers increases as the walk evolves in time. At a time, t, a characteristic scale for the velocity experienced by the walkers is $|u(t)| \sim y(t)^{\beta} \sim (D_y t)^{\beta/2}$. Accordingly, the characteristic x_{rms} displacement should scale as $x_{rms} \sim t \cdot |u(t)| \sim (D_y t)^{\beta/2}\, t$ and thus, $\sigma_x \sim t^{1+\beta/2}$.

The authors then derived exact solutions for the distributions, $p(x,t)$, for the longitudinal displacement of the random walk on the shear flow $|u(y)| = u_o y^{\beta}$ for the cases $\beta = 0, 1$. The case $\beta = 0$, corresponds to a case of split flow (Fig. 1), where the magnitude of u is constant, but in opposing directions above and below the $x-$axis. In this case, the displacement for a single walker is simply $n^+ - n^-$, where n^+ is the number of steps in the walk spent above the axis, in the positive u region, and n^- is the number of steps spent below the x-axis. The problem therefore reduces to finding the probability to spend n steps on one side of the origin or the other in a 1-D random walk, which is the well-known arcsine distribution. The authors define the dimensionless variables $\xi = x/\left(u_o t\right)(Dt)^{\beta/2}$and $\eta = y/\sqrt{Dt}$ and the distribution for longitudinal position is obtained: $p(\xi) \sim \frac{1}{\pi(1-\xi^2)^{1/2}}$, (the arcsine distribution) (Fig. 2). This is an unusual distribution because it has its minimum at the origin and $p(\xi) \to \infty$ for $\xi \to 1$. The most likely path for a walker is therefore on one side of the origin or the other, despite the fact that $\langle u(y) \rangle = 0$.

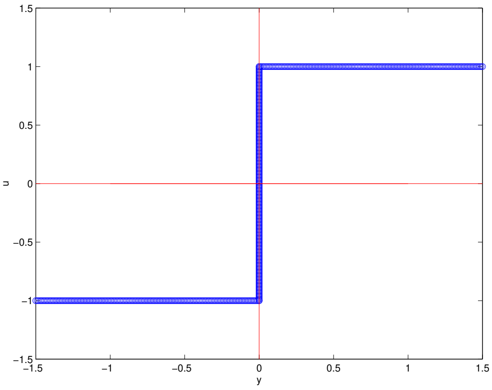

Fig. 1 Split Flow, $|u(y)| = u_o$

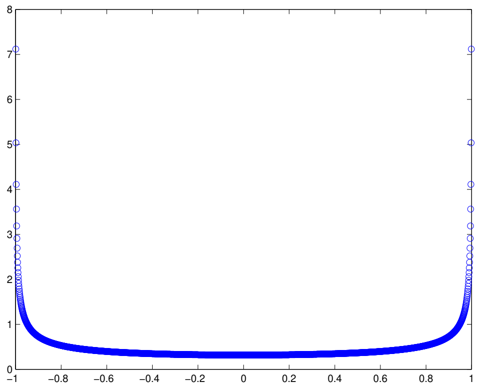

Fig. 2 $P(\varsigma)$

For a linear shear profile, $|u(y)| = u_o y$ (Fig. 3), the authors find the limiting 2-D distribution after n steps, $P_n(x, y)$. For this walk, the longitudinal displacement increases linearly with n, so the walk is a simple walk with a linearly increasing step size. The resulting distribution is $P_n(x, y) \sim \frac{\sqrt{3}}{\pi u_o n^2} e^{(-y^2/2n)} e^{\left(-6(x - \langle u \rangle n)^2 / u_o^2 n^3\right)}$ where $\langle u \rangle = u_o y/2$ (Fig.4). This describes a Gaussian profile that scales anomalously with n : $\sigma^2 \sim n^3$. Note the anisotropy between σ_x and σ_y in the contour plot (Fig. 4). Generally speaking , the longitudinal dispersion is almost always much greater than the transverse diffusion in shear flows.

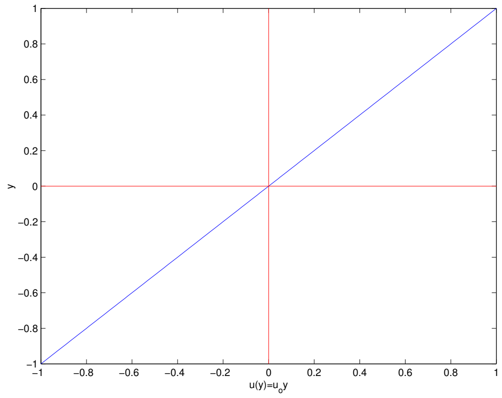

Fig. 3 Linear Shear, $|u(y)| = u_o y$

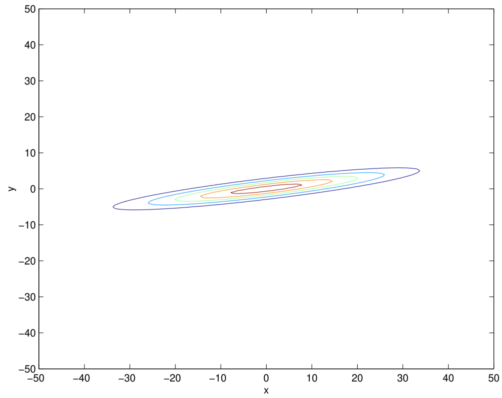

Fig. 4 Contour Plot of $P(x, y)$

1.4 Shear Dispersion in Random Velocity Fields

Another interesting case of shear dispersion in an unbounded flow field is that for which the velocity field, $u(y)$, is a random function of y (Fig. 5). For this case, both the transverse and longitudinal displacements are random functions. However, the random profile, $u(y)$ is still constant in x. Thus, the system conceptually consists of an infinite number of stratified layers, each with a constant, but randomly assigned velocity, and the only movement between the layers is by way of the transverse diffusion process.

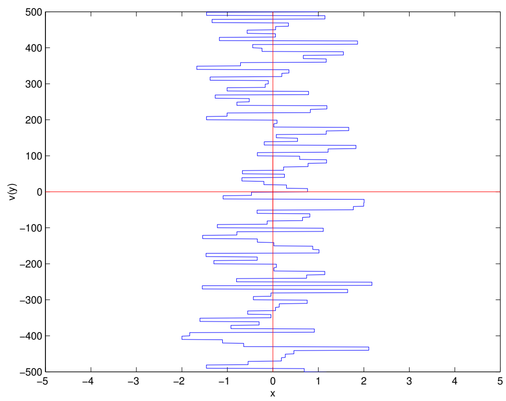

Fig. 5 A Representative Random Velocity Profile, $u(y)$

It seems that Matheron and de Marsily [4] were the first to study this problem, and did so in the context of flow in layered porous media. They introduced an integrated Langevin equation to describe the random displacements:

$$
\begin{aligned}
x(t) &= x_o + \xi(t) + \int_0^t u(y(t))dt \\
y(t) &= y_o + \zeta(t)
\end{aligned}
\tag{1.6}
$$

where $\xi(t)$ describes the random walk (Brownian motion) in the longitudinal direction and $\zeta(t)$ describes the transverse random motion. The integral, $\int_0^t u(y(t))dt$, is a random integral containing all of the details of the transverse diffusion process, i.e. a sum over all of the random paths. The authors found the expected variance, $\langle x(t)^2 \rangle$, by arduous integrations and found anomalous scaling for the longitudinal variance, $\sigma^2 \sim t^{3/2}$.

The same problem was studied by Redner [5] who found the same scaling for the longitudinal variance, and gave a heuristic argument for the anomalous scaling. His argument is that although the mean velocity averaged over an infinite number of layers in a particular velocity profile, $\langle u(y) \rangle$, is zero, the number of layers visited by a single random walker is always finite. An average of the velocity over the number of layers visited is a finite fluctuating quantity and thus there exists a non-vanishing bias that results in superdiffusive scaling.

An exact expression for σ_x^2 in this problem was obtained by Bouchaud et. al. [2] They also began with the Langevin equation of motion for the random walk:

$$\begin{aligned}\frac{dx}{dt} &= u(y(t)) \\ \frac{dy}{dt} &= \eta(t)\end{aligned} \tag{1.7}$$

They integrated the first equation to obtain an expression for $x(t)$,

$$x(t) = \int_0^t u(y(t'))dt' = \int_{-\infty}^{+\infty} \eta(y,t)u(y)dy \tag{1.8}$$

where $\eta(y,t)$ is the total time spent per layer in each layer and scales as the total elapsed time, t,divided by the number of layers visited which scales as $\sqrt{Dt}$ so $\eta(y,t) \sim \frac{\sqrt{t}}{\sqrt{D}}$.

The moments of the distribution for longitudinal position were then given by

$$\langle\langle x^n(t)\rangle_w\rangle_c n! \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_{n-1}} dt_n \, \langle\langle u(y(t_1))\cdots u(y(t_n))\rangle_w\rangle_c \tag{1.9}$$

where the double brackets imply first averaging over all random transverse trajectories, w, and then averaging over all possible ensembles of random velocity profiles, c. The velocity correlation function inside the integral was given to be

$$\begin{aligned}\langle\langle u(y(t_1))\cdots u(y(t_n))\rangle_w\rangle_c = &\int_{-\infty}^{+\infty} dy_1 dy_2\cdots dy_n \, \langle u\,(y_1)\cdots u\,(y_n)\rangle_c \\ &\times p(y_n,t_n)p(y_{n-1}-y_n,t_{n-1}-t_n)\cdots p(y_1-y_2,t_1-t_2)\end{aligned} \tag{1.10}$$

where $p(y,t)$ is the Gaussian propagator for the transverse walk, $p(y,t) = \frac{1}{\sqrt{4\pi Dt}}e^{(-y^2/4Dt)}$. Thus in computing the velocity correlation, the averaging over c and w are can be performed separately. The product of Gaussians in (1.10) is the probability that the walker visits the sequence of positions in the integration at times, t_i. For the random velocity configuration studied by the authors, the spatial correlation $\langle u(y_1)\cdots u(y_n)\rangle_c$ is a product of δ functions with standard deviation σ and so they found the second moment to be

$$\left\langle\left\langle x^2(t)\right\rangle_w\right\rangle_c = 2\sigma\int_0^t dt_1\int_0^{t_1} dt_2\int_{-\infty}^{+\infty} dyp(0,t_1-t_2)p(y,t_2) = \frac{4\sigma}{3\sqrt{\pi D}}t^{3/2}$$

which shows the anomalous super-diffusive scaling for the variance, $\sigma_x \sim t^\nu \;\; \nu = 3/4$.

The authors note that though the mean displacement, $\langle\langle x(t)\rangle_w\rangle_c$ averaged over all walks for all possible velocity profiles is zero, the mean averaged over all walks for a given random profile, $\langle x(t)\rangle_w$ is often nonzero even for large t. This seemingly would make it quite difficult to make accurate predictions about a particular system, since all that is known are the statistics of all possible systems.

Chapter 2

Analysis of Shear Dispersion for a Sinusoidal Velocity Profile

2.1 Presentation of the Problem

For the second part of the project, I applied some of the principles elucidated from the study of random walk shear dispersion to determine the longitudinal variance, σ_x^2 expected from a sinusoidally varying velocity profile. For this case, the velocity is given by $v(y) = v_o \sin(\frac{2\pi}{L} y)$, and is steady in the longitudinal direction (Fig. 6). A simple random walk with diffusion constant D_y takes place transverse to the velocity field.

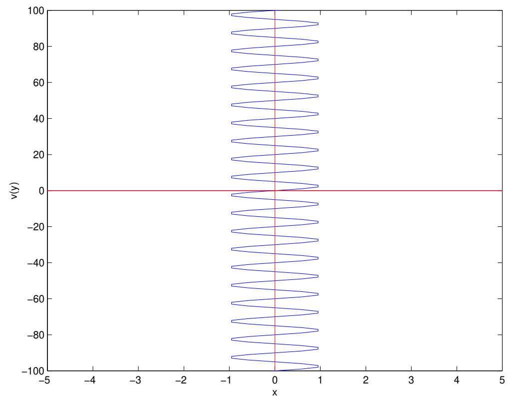

Fig. 6 Sinusoidal Velocity Profile, $v(y) = v_o \sin(\frac{2\pi}{L} y)$

I was interested in this analysis because I wanted to examine the relationship between the scale of the spatial fluctuations of $u(y)$ and the effective longitudinal diffusion. This has application in my research, which concerns the dispersion of tracer particles in fluid flow through plant canopies. In such flows, the plants are modeled as cylinders randomly distributed through the flow domain which consists primarily of a uni-directional current that may or may not be fully turbulent. A typical velocity profile within the canopy is shown in Fig. 7.

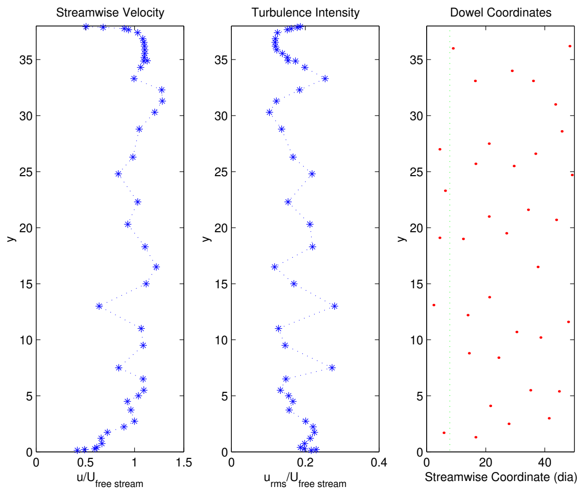

Fig. 7 Typical Velocity Profile in Plant Canopy, with Turbulence Intensity and Cylinder Coordinates

One primary mechanism for dispersion is the trapping and slow release of tracer from recirculating vortices behind cylinders, which can be modeled from the perspective of continuous time random walks as a waiting time distribution. A second process, and the one which is relevant for this project topic is shear dispersion, which occurs due to the inhomogeneity of the flow field caused by the random cylinder arrangement. The presence of the cylinders creates layers of varying velocity, $u(y)$, that are of the order of the mean cylinder spacing in width. Thus it is desired to know the functional relationship between σ_x, the layer width, and transverse diffusion.

2.2 Dimensional Analysis

The results for the sinusoidally varying velocity profile, $v(y) = v_o \sin(\frac{2\pi}{L}y)$ may shed some light on the functional relationship that is sought. In this system, the relevant parameters are v_o, magnitude of the velocity, L, the wavelength of the oscillations, and D_y, the transverse diffusion constant. If it is assumed that the resulting longitudinal dispersion process will be normally diffusive, that is $\sigma_x \sim t^{1/2}$, then it is appropriate to define an effective longitudinal dispersion constant, D_x. It turns out that results of the numerical simulation will support this assumption. Then, the four parameters, v_o, L, D_y, and D_x should completely describe the system.

Doing a dimensional analysis of these four variables, with D_x the dependent variable, leads to two dimensionless groups:

$$\Psi \equiv \frac{D_x}{v_o L} \qquad \Phi \equiv \frac{v_o L}{D_y} \tag{2.1}$$

That these two dimensionless groups are reasonable is demonstrated by the following argument. The characteristic longitudinal displacement, σ_x, should increase in proportion to both the magnitude of the velocity, v_o, and the characteristic time for a random walker to remain in a region of constant velocity, wherein its longitudinal displacement is correlated. A reasonable such timescale is the time for a walker to traverse a single period of the sinusoidal velocity profile, $\frac{L^2}{D_y}$. Thus $\sigma_x \sim \frac{L^2}{D_y} v_o$ and therefore $D_x = \frac{1}{2}\frac{d\sigma_x^2}{dt} \sim \frac{\sigma_x^2}{L^2/D_y} \sim \frac{\left(v_o L^2/D_y\right)^2}{L^2/D_y} \sim \frac{v_o^2 L^2}{D_y}$. This scaling would suggest the dimensionless groups Ψ and Φ as written.

2.3 Numerical Simulations

Numerical Simulations were conducted with a random walk model that follows the motion of the walkers in two dimensions. The equations governing the motion are the Langevin equations,

$$\begin{aligned} \frac{dx}{dt} &= u(y(t)) = v_o \sin(\frac{2\pi}{L} y(t)) \\ \frac{dy}{dt} &= \xi(t) \end{aligned} \tag{2.2}$$

The random process, $\xi(t)$, characterizes the simple random walk in y with Gaussian transition probability. In discretized form the, single step displacements are

$$\begin{aligned} \Delta x &= v(t) \cdot \tau = \tau v_o \sin(\frac{2\pi}{L} y(t)) \\ \Delta y &= \sqrt{2 D_y \tau} \cdot \varsigma \end{aligned} \tag{2.3}$$

where τ is the time for a single step, the discrete time step in the numerical model, and ς is a random variable with a standard Gaussian distribution. The expression for Δy ensures a Gaussian transition probability with $\sigma = \sqrt{2D_y\tau}$.

The simulations were carried out on a 2-D grid, with a spatial resolution of $L/10$ and timestep, $\tau = \frac{1}{10}\frac{L^2}{D_y}$, $1/10$ the timescale to traverse a single velocity period. The number of walkers was 1000 for each case. These discretizations as well as the finite number of walkers account for the sources of numerical error, since the positions of the walkers are precisely known at each timestep and are not subject to discretization. The duration of each simulation was $100^2\frac{L^2}{D_y}$, or the timescale for the transverse walk to traverse a transverse distance of $100L$. The random component of the y displacement, ς, was determined for each walker at each timestep by choosing a random sample from a distribution of normally-distributed numbers (mean zero and $\sigma = 1$) of size 10^6 (Fig. 8).

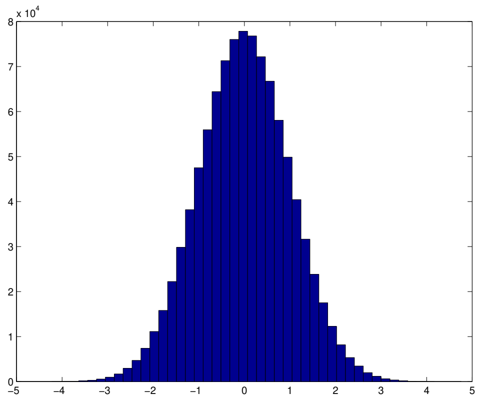

Fig. 8 Histogram plot of the Gaussian distribution randomly sampled to determine single step size, Δy

As a verification of the numerical model, a simulation was done in the absence of a velocity profile, so that the only motion was the 1-D random walk. The results for $P(\varsigma)$, shown in Fig. 9 below, verify that the Gaussian behavior is accurately obtained.

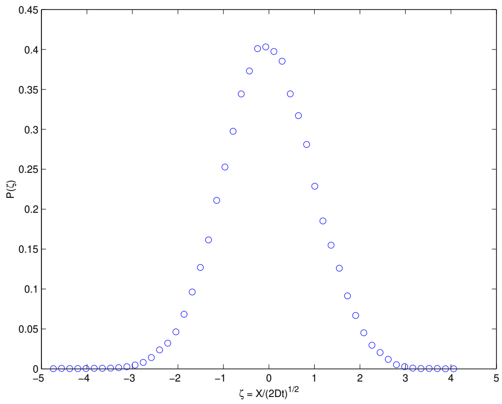

Fig. 9 Displacement from the origin, $P(\varsigma)$ for a 1-D random walk averaged over all times during the walk. t=10,000 steps.

2.4 Simulation Results

Simulations were carried out for a range of values of the dimensionless variable Φ from 10^{-2} to 10^{3}. This was done by varying L, D_y, and v_o all in equal measure. The results show a clear linear relationship between Ψ and Φ,

$$\Psi = \alpha\Phi \tag{2.4}$$

where α is a constant, thus signifying normal diffusive scaling by affirming that the choice of a D_x was reasonable.

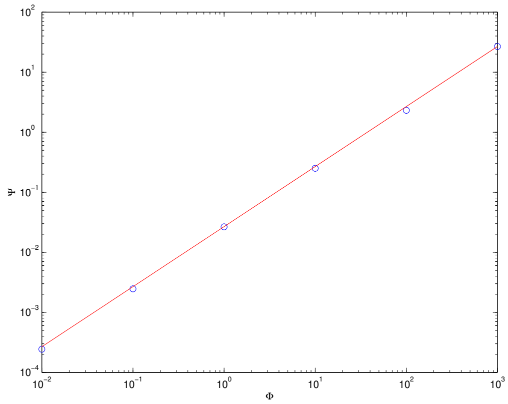

Fig. 10 Loglog plot of Ψ vs. Φ (open circles) with line of slope = 0.026

Figure 10 shows a loglog plot of Ψ vs. Φ, which demonstrates a linear relationship with slope = 1, signifying the linear dependence of Ψ on Φ. The linear scaling of is also shown by Fig. 11, a plot of σ_x^2 vs. t for a representative simulation.

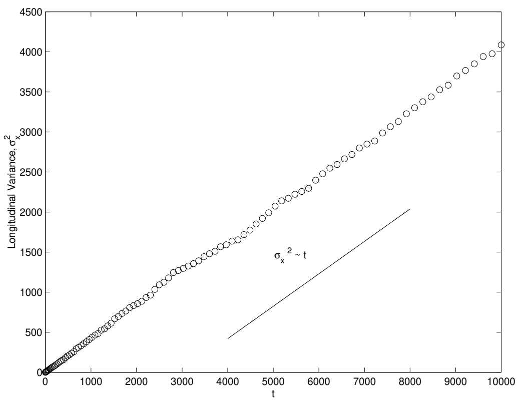

Fig. 11 Longitudinal variance, σ_x^2 vs. t for the simulation $\Psi = 1$

The constant, α can be determined by a linear least squares fit of the data, but since the data are distributed across decades, the fitting would be biased toward large values of Ψ. As an alternative, α was obtained by simply averaging the dimensionless ratio, Ψ/Φ, over all six simulations. The result is

$$\alpha = 0.025 \pm 0.0013 \tag{2.5}$$

and it follows that $D_x \simeq \frac{1}{40}\frac{v_o^2 L^2}{D_y}$ and $\sigma_x^2 = 2D_x t \simeq \frac{1}{20}\frac{v_o^2 L^2}{D_y} t$.

Ψ vs. Φ for each simulation

Φ	Ψ
0.01	0.00024324
0.1	0.002464
1.0	0.02661
10	0.25073
100	2.3146
1000	26.731

The scaling of the second moment signifies that this system is governed by normal diffusive scaling, but the asymptotics of the dispersion process are yet unknown. To get an idea of the time necessary to reach the asymptotic dispersion process, the distribution, $P(\varsigma)$, where $\varsigma = \frac{x}{\sqrt{2D_x t}}$, is examined for fixed ς and small t. Fig. 12 shows plots of $P(\varsigma)$ for a single simulation for early times. By the time $t = 3\frac{L^2}{D}$, $P(x,t)$ very much resembles the standard

Gaussian scaling, $P(\varsigma) \sim \frac{1}{\sqrt{2\pi}} e^{(-\varsigma^2/2)}$.

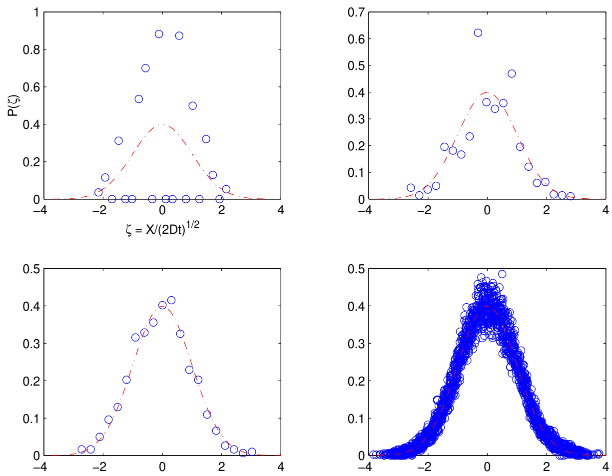

Fig. 12 The approach to Gaussian limiting behavior. The first three plots show $P(\varsigma)$ for $t = 1\frac{L^2}{D_y}, 2\frac{L^2}{D_y}, 3\frac{L^2}{D_y}$. The last plot shows $P(\varsigma)$ for all times thereafter.

Furthermore, the kurtosis, a good measure of deviation from the Gaussian, approximately reaches the Gaussian value of 3 by about $t = 3\frac{L^2}{D_y}$ and continues to hover around that value, with some fluctuations, for the duration of the walk (Fig. 13). Therefore, it seems that a good estimate for the asymptotic time limit is around $t \simeq 3\frac{L^2}{D_y}$.

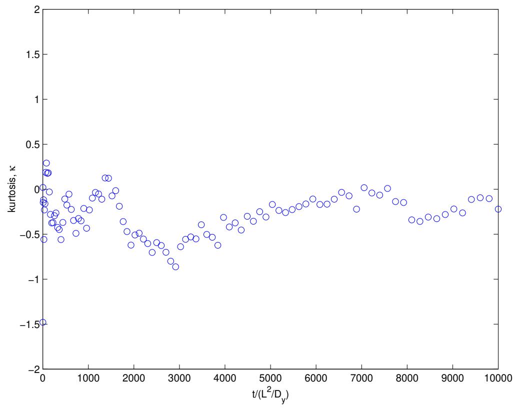

Fig. 13 Kurtosis, κ, vs. t over one simulation

The very well-behaved Gaussian scaling of $P(\varsigma)$ makes sense when one considers the conditions necessary for the asymptotic dispersion limit to hold. The random walkers must sample all of the velocities in the system so that $\langle u(t) \rangle_w$, the average velocity experienced by all walkers, approaches zero. This condition is certainly satisfied relatively quickly in the sinusoidal profile since the velocities experienced by a walker repeat themselves over the trajectory of the transverse walk. One might conjecture that it would require a traverse of only a few periods for the average velocity experienced over the course of a walk to become zero and for the walker to begin experiencing the same longitudinal displacements that were previously experienced. Indeed that is about the time necessary ($t \simeq 3\frac{L^2}{D_y}$).

The results from the analysis of the sinusoidally varying profile show that $\sigma^2 \sim L^2$. This functional relationship suggests that the dispersion increases with the square of the length scale over which the velocity varies in a particular shear profile. This may only hold for the case of normally diffusive processes, since, when anomalous scaling occurs, the dimensionless variables from the above analysis change: there is no effective diffusion constant, D_x and a different scaling relationship exists between σ and t. But assuming the process is governed by normal diffusion, the results of this analysis seem to have accomplished their desired end, suggesting a relationship between the mean cylinder spacing and D_x in the plant canopy.

A more refined analysis might be to model the plant canopy as layers of randomly distributed velocity, $u(y)$, similar to the problem of random shear flows discussed above. However, the difference between this case and the one previously discussed is that within the plant canopy, the velocity in the layers varies with longitudinal position as well, $u = u(x, y)$ where u is a random variable.

Chapter 3

Summary

For this project, the phenomenon of shear dispersion was studied from the perspective of random walks. For Brownian particles undertaking a simple random walk transverse to a spatially varying current, a spectrum of different behavior can result. In the well-behaved limit, when the walkers are able to experience all representative velocities in a system, after an asymptotic time is reached, the growth of longitudinal variance exhibits normal diffusive scaling and can be characterized by an effective dispersion coefficient, as G.I. Taylor showed.

On the other hand, when the velocity profile exhibits power law behavior such that progressively larger x displacements are experienced as the walk evolves, superdiffusive scaling results, with σ_x growing faster than $t^{1/2}$. Another instance of anomalous diffusion was seen to result when the velocity profile is a random function, $u(y)$, so that, while the mean velocity over the infinite profile becomes zero, individual random walkers always have a finite, but fluctuating mean drift which does not vanish, and so super-diffusion again results.

These results stimulated the study of a velocity profile that fluctuated in a very predictable way: a sinusoidal function. A dimensional analysis together with numerical simulations yielded an expression for $\langle x^2 \rangle$. The process was found to be normally diffusive, and in fact to reach the asymptotic Gaussian limit after relatively short times. This was explained by the fact that due to the periodicity, the walkers re-experience the same velocities continually and so they very quickly lose any memory, or velocity correlation effects.

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