Lecture 9: Examples of Random Walks

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In the last lecture, the formulas for the exact solutions of random walks were presented, but these solutions are more useful if the integrals are solvable. The solution for the gaussian random walk was shown previously, and in this lecture, the procedure will be further exemplified using a simple random walk on the integers, a leptokurtotic ($\kappa > 1$) function with all moments finite and another leptokurtotic function with divergent moments of order higher than 2.

1 Example 1: Simple walk

The simple random walk on the integers has the transition probability

$$p(x) = \frac{1}{2}[\delta(x-\sigma) + \delta(x+\sigma)] \tag{1}$$

from which the structure function can be determined by

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{-ikx} p(x) dx \tag{2}$$

$$= \frac{1}{2}[e^{ik\sigma} + e^{-ik\sigma}] = \cos k\sigma \tag{3}$$

$$\sim 1 - \frac{\sigma^2 k^2}{2!} + \frac{\sigma^4 k^4}{4!} - O(k^6) \tag{4}$$

and its logarithm is

$$\log \hat{p}(k) = \log \cos k\sigma = \log\left(1 - \frac{\sigma^2 k^2}{2!} + \frac{\sigma^4 k^4}{4!} - \cdots\right) \tag{5}$$

which can be developed using $\log(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \cdots$ as

$$\log \hat{p}(k) = -\frac{\sigma^2 k^2}{2!} + \left(\frac{1}{4!} - \frac{1}{8}\right)\sigma^4 k^4 + \cdots \tag{6}$$

From the Taylor expansion (4), the moments can be obtained:

$$m_2 = \sigma^2 \tag{7}$$
$$m_4 = \sigma^4 \tag{8}$$
$$m_6 = \cdots \tag{9}$$

and similarly from (6) the cumulants can be determined:

$$c_2 = \sigma^2 \tag{10}$$
$$c_4 = \sigma^4 \left(1 - \frac{4!}{8}\right) = -2\sigma^4 \tag{11}$$
$$c_6 = \cdots \tag{12}$$

The exact solution for the probability distribution after n steps can be obtained convolving the transition probability with itself n times

$$P_n(x) = p(x)^{*n} \tag{13}$$

an operation which is translated into a power in Fourier space

$$\hat{P}_n(k) = \hat{p}(k)^n = \cos^n(k\sigma) \tag{14}$$

from which the distribution may be found reversing the transform

$$P_n(x) = \int_{-\infty}^{\infty} e^{ikx} \cos^n k\sigma \frac{dk}{2\pi} \tag{15}$$
$$= \sum_{m=0}^{n} \binom{n}{m} \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m} \delta[x - \sigma(2m-n)] \tag{16}$$

This expression can be interpreted as the sum of all combinations of m steps to the right and $n-m$ steps to left, giving a final position at $\sigma(2m-n)$, multiplied by their probabilities.

2 Example 2: Leptokurtotic function with finite moments

The next example is the random walk with transition probability

$$p(x) = \frac{A}{a + \cosh(x)} \tag{17}$$

which is exponentially wider than a gaussian (is roughly $\sim Ae^{-|x|}$ as $x \to \infty$), but still has all moments finite. The normalization factor A will be determined later.

The structure function is

$$\hat{p}(k) = A \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{a + \cosh x} dx \tag{18}$$

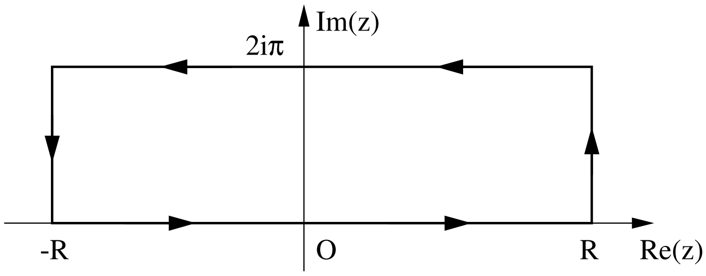

Figure 1: Integration contour for Example 2.

which can be solved by contour integration of $f(z) = \frac{e^{-ikz}}{a+coshz}$ along the path shown in Figure 1.

The integral along the lower line is $\hat{p}(k)$, and along the upper line is $-e^{2k\pi}\hat{p}(k)$, because $\cosh z$ is periodic in the imaginary direction with period $2\pi i$. Since $f(z)$ at the lateral parts is bounded by

$$\left| \int \frac{e^{-ikz}}{a+\cosh z} dz \right| \leq \int \left| \frac{e^{-ikz}}{a+\cosh z} \right| dz \leq 2\pi i \frac{e^{-ikR}e^{2\pi k}}{\frac{e^R+e^{-R}}{2} - a} \tag{19}$$

it is seen that the contributions on these parts become negligible as $R \to \pm\infty$.

The final expression is

$$\oint f(z)dz = \frac{1}{A}\hat{p}(k)(1 - e^{2k\pi}) = 2\pi i \sum_i \operatorname{Res}(\alpha_i) \tag{20}$$

where $\operatorname{Res}(\alpha_i)$ are the residues at the poles α_i of $f(z)$ contained in the contour, or the points where

$$a + \cosh \alpha_i = 0 \tag{21}$$

$$\cosh \alpha_i = -a \tag{22}$$

or, using the $\cosh(x)$ definition,

$$\frac{e^{\alpha_i} + e^{-\alpha_i}}{2} = -a \tag{23}$$

Defining $\omega = e^{\alpha_i}$ and multiplying both sides by ω,

$$\omega^2 + 1 = -2a\omega \tag{24}$$

$$\omega = -a + \sqrt{a^2 - 1} \tag{25}$$

therefore

$$\alpha_i = \log(-a + \sqrt{a^2 - 1}) \tag{26}$$

For different ranges of a values, different strategies can be used in determining α_i. These strategies can be shown to yield the same expression for the residues $\operatorname{Res}(\alpha_i)$. The case

$|a| < 1$ will be detailed as an example. There will be two imaginary values $\alpha_1 = i\pi - i\beta$ and $\alpha_2 = i\pi + i\beta$, where β is a real number such that $\cosh(i\pi \pm i\beta) = \cos(\pi \pm \beta) = -\cos\beta = -a$, that is, $\beta = \cos^{-1} a$.

The residues are

$$\mathrm{Res}(\alpha_i) = \frac{e^{-ik\alpha_i}}{\sinh \alpha_i} \tag{27}$$

$$= \frac{e^{\pm k\beta} e^{k\pi}}{i \sin(\pi \pm \beta)} \tag{28}$$

therefore, from (20),

$$\hat{p}(k) = 2\pi i A e^{k\pi} \left(\frac{e^{-k\beta}}{i \sin\beta} - \frac{e^{k\beta}}{i \sin\beta} \right) \frac{1}{1 - e^{2k\pi}} = 2\pi A \left(\frac{\sinh k\beta}{\sin\beta \sinh k\pi} \right) \tag{29}$$

The normalization factor can be obtained from

$$\int_{-\infty}^{\infty} p(x) dx = 1 \tag{30}$$

$$\hat{p}(0) = 2A \frac{\beta}{\sin\beta} = 1 \tag{31}$$

$$A = \frac{\sin\beta}{2\beta} \tag{32}$$

where $\hat{p}(0)$ was determined using L'Hopital rule.

This function has a Taylor expansion

$$\hat{p}(k) \sim 1 - m_2 \frac{k^2}{2!} + m_4 \frac{k^4}{4!} - \cdots \tag{33}$$

from where the moments can be determined as usual:

$$m_2 = \frac{1}{3}(\pi^2 - \beta^2) \tag{34}$$

$$m_4 = \frac{1}{5} m_2 (3\beta^2 - 7\pi^2) \tag{35}$$

$$m_6 = \frac{1}{7} m_2 (3\beta^4 - 18\pi^2\beta^2 + 31\pi^4) \tag{36}$$

And its logarithm also has a Taylor expansion

$$\log \hat{p}(k) \sim -c_2 \frac{k^2}{2!} + c_4 \frac{k^4}{4!} - \cdots \tag{37}$$

and the cumulants are

$$c_2 = m_2 \tag{38}$$

$$c_4 = -\frac{2}{5} m_2 (\beta^2 + \pi^2) \tag{39}$$

$$c_6 = -\frac{16}{21} m_2 (\beta^2 + \pi\beta + \pi^2)(\beta^2 - \pi\beta + \pi^2) \tag{40}$$

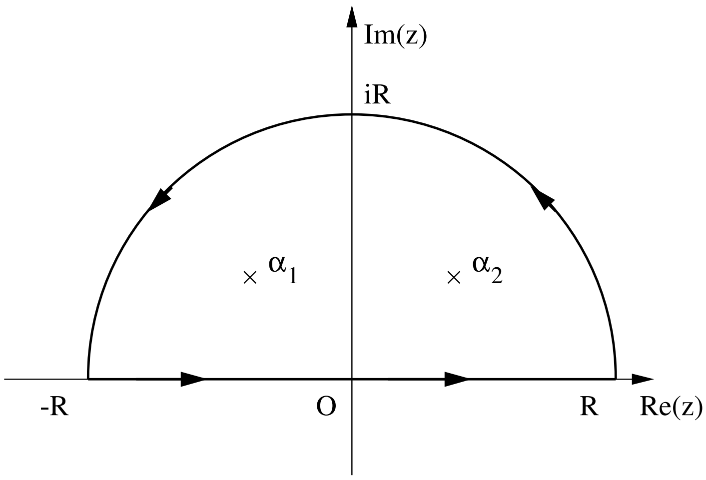

Figure 2: Integration contour for Example 3.

Finally, these cumulants may be used to determine the probability distribution after n steps, as shown in the previous lecture. After the inverse transform,

$$P_n(\zeta) = \frac{e^{-\zeta/2}}{\sqrt{2\pi n}}\left(1 + \frac{c_4 f_1(\zeta)}{n} + \cdots\right) \tag{41}$$

where $\zeta = x/\sqrt{n}$, which shows that this distribution obeys the Central Limit Theorem.

3 Example 3: Leptokurtotic function with only the second moment finite

The transition probability

$$p(x) = \frac{A}{1+x^4} \tag{42}$$

has very fat tails, which cause its moments of order higher than 2 to diverge (including $\langle |x|^3 \rangle$, although $\langle x^3 \rangle = 0$ because of symmetry). Its structure function is

$$\hat{p}(k) = \int_{-\infty}^{\infty} A \frac{e^{-ikx}}{1+x^4} dx \tag{43}$$

and its form will again be sought by contour integration, using the contour on Figure 2.

First, the integral on the radial contour $|z| = R$ will shown to be zero, using the inequality

$$\left|\int f(z)dz\right| \leq \int |f(z)|dz \tag{44}$$

$$\left|\int \frac{e^{-ikz}}{1+z^4} dz\right| \leq \pi R \frac{1}{R^4 - 1} \tag{45}$$

where the fact that $R^4 - 1$ is a lower bound to $|z^4 - 1|$ was used. As $R \to \infty$ with $k < 0$, this ensures that $f(z) \to 0$ in the radial contour, therefore the contour integral is simply

$$\hat{p}(k) = 2\pi i[\mathrm{Res}(\alpha_1) + \mathrm{Res}(\alpha_2)] \tag{46}$$

where the poles α_1 and α_2, indicated in Figure 2, are the roots of $1 + x^4 = 0$ with positive complex part, that is $\alpha_1 = \frac{-1+i}{\sqrt{2}} = e^{3i\pi/4}$ and $\alpha_1 = \frac{1+i}{\sqrt{2}} = e^{i\pi/4}$. Therefore

$$\hat{p}(k) = 2\pi i A\left(\frac{e^{-ik(1+i)/\sqrt{2}}}{4e^{3\pi i/4}} + \frac{e^{-ik(-1+i)/\sqrt{2}}}{4e^{9\pi i/4}}\right) \tag{47}$$

$$= \frac{\pi}{2}A\frac{e^{k/\sqrt{2}}}{e^{i\pi/4}}\left((1+i)\cos\frac{k}{\sqrt{2}} - (1+i)\sin\frac{k}{\sqrt{2}}\right) \tag{48}$$

where both forms for α_i were used. Replacing $e^{i\pi/4}$ with the alternative $(1+i)/\sqrt{2}$,

$$\hat{p}(k) = \frac{\pi}{\sqrt{2}}Ae^{k/\sqrt{2}}\left(\cos\frac{k}{\sqrt{2}} - \sin\frac{k}{\sqrt{2}}\right) \tag{49}$$

Recalling that the transform of an even function must itself be a real even function, then the formula for all k may be obtained from the particular case $k < 0$ above. Also, making $\hat{p}(0) = 1$, it is found that $A = \sqrt{2}/\pi$, therefore

$$\hat{p}(k) = e^{|k|/\sqrt{2}}\left(\cos\frac{k}{\sqrt{2}} + \sin\frac{|k|}{\sqrt{2}}\right) \tag{50}$$

Now, the difference to the previous cases is that this function does not have a Taylor expansion, since its moments $\langle|x|^\mu\rangle$ diverge for $\mu > 2$. But there is an expansion

$$\hat{p}(k) \sim \left(1 - \frac{|k|}{\sqrt{2}} + \frac{k^2}{4} - \cdots\right)\left(1 - \frac{k^2}{4} + \frac{k^4}{4\cdot 4!} - \cdots + \frac{|k|}{\sqrt{2}} - \frac{|k|^3}{12\sqrt{2}} + \cdots\right) \tag{51}$$

$$\sim 1 - \frac{k^2}{2} + \frac{|k|^3}{2\sqrt{2}} - \frac{5k^4}{48} + O(|k|^5) \tag{52}$$

where the second term on (52) correspond to the gaussian behavior and the third term signals that the 3rd and higher moments diverge. Similarly, the cumulants may be obtained as

$$\log\hat{p}(k) \sim -\frac{k^2}{2} + \frac{|k|^3}{3\sqrt{2}} - \frac{11k^4}{48} + \cdots \tag{53}$$

where the same observations may be made. The fourth-order term is "not quite" the kurtosis, since the fourth moment diverge.

The probability density for n steps can be obtained as usual from the cumulants using

$$P_n(x) = \int_{-\infty}^{\infty} e^{ikx}\hat{p}(k)^n\frac{dk}{2\pi} = \int_{-\infty}^{\infty} e^{ikx}e^{-k^2n/2}\left(e^{|k|^3n/3\sqrt{2}-\cdots}\right)\frac{dk}{2\pi} \tag{54}$$

or, using the variables $\omega = k\sqrt{n}$ and $\zeta = \frac{x}{\sqrt{n}}$ (from which $kx = \omega\zeta$),

$$P_n(\zeta) = \int_{-\infty}^{\infty} e^{i\omega\zeta}e^{-\omega^2/2}\left(e^{|\omega|^3/3\sqrt{2n}-\cdots}\right)\frac{d\omega}{2\pi} \tag{55}$$

which suggests that again the central limit theorem holds, even though the moments are not available. In the next lecture, the integrals will be evaluated in order to prove this fact.