Lecture 8: Exact Solutions for Random Walks

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1 Convolutions and Addition of Random Variables

Consider a discrete random walk[1] with a transition probability given by $p_n(x)$, where n is the step number and x is the step. In terms of the flight as a whole, this is the probability density function of ΔX_n, the displacement of the nth step.

The probability density function (PDF) for the position of the walker after N steps will be denoted $P_N(x)$, and the position after N steps will be given by

$$X_n = \sum_{n=1}^{N} \Delta x_n$$

Using Bachelier's equation, we can express P_i in terms of the p_j

$$\begin{aligned} P_2(x) &= \int_{-\infty}^{\infty} p_2(x-y)p_1(y)dy \\ P_2(x) &= (p_2 * p1)(x) \\ P_3(x) &= \int_{-\infty}^{\infty} p_3(x-y)(p_2 * p_1)(y)dy = (p_3 * p_2 * p_1)(x) \end{aligned}$$

where the $*$ denotes the **convolution**. If the steps are identical, $P_n(x) = p_1^{*n}(x)$, with the notation of the **convolution power**. More generally though, we have

$$P_n(x) = (p_n * p_{n-1} * \ldots * p_2 * p_1)(x)$$

Notice that here by using Bachelier's equation, we have assumed that the steps are *independent* (Markov's hypothesis). Note also that the order in which we write the convolution does not matter. Convolutions are also linear. In fact, in the space of functions which are square integrable, $\mathcal{L}^{\in}$, the convolution product forms a ring. This follows from the correspondence with Fourier space, the spaces are isomorphic, and from the fact that in Fourier space convolutions are simple products.

[1] In this lecture we are considering the position of the walker after n steps, so it is makes no difference whether we speak of "random walks" or "random flights". Recall that in the latter case each step takes one unit of time, while in the former case the walker moves at constant velocity so that the time for step n is proportional to the random quantity $|\Delta X_n|$.

2 The Structure Function and Moments

We now introduce the structure function of the flight, also called characteristic function, to be the Fourier Transform of the transition PDF.

$$\hat{p}_n(k) = \int_{-\infty}^{\infty} e^{-ikx} p_n(x)dx$$
$$p_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{p}_n(x)dk$$

Here the inverse transform is also given. The Fourier transform is a very useful tool in the context of discrete random walks. It provides an easy way to compute the moments of the transition PDF.

$$m_{n,l} = \int_{-\infty}^{\infty} x^l p_n(x)dx = i^l \left.\frac{d\hat{p}_n(k)}{dk^l}\right|_{k=0}$$

Here $m_{n,l}$ stands for the lth moment of the nth step. This is found by differentiating the expression for $\hat{p}_n(k)$. Using this, it is easy to write the Taylor series for the structure function in terms of the moments of the transition PDF.

$$\begin{aligned}\hat{p}_n(k) &= \sum_{l=0}^{\infty}\frac{(-ik)^l}{l!}m_{n,l}\\ \hat{p}_n(k) &= \sum_{l=0}^{\infty}(ik)^l D_{n,l}\tau\end{aligned}$$

where we have reintroduced the notation $D_{n,l} = m_{n,l}/(\tau l!)$.

3 The Convolution Theorem

The most important property of the Fourier transform though is that is converts convolutions into simple multiplications. More precisely

$$\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k).$$

The idea of the proof is the following:

$$\begin{aligned}\widehat{f * g}(k) &= \int_{-\infty}^{\infty} dx e^{-ikx}\int_{-\infty}^{\infty} dy f(x-y)g(y)\\ &= \int_{-\infty}^{\infty} dy g(y)\int_{-\infty}^{\infty} dx f(x-y)e^{-ikx}\end{aligned}$$

where we have inverted the order of integration. This step would require more justification, but as long as both integrals are absolutely convergent, it does not cause any problems. If we now let $z = x - y$ we find

$$\widehat{f * g}(k) = \int_{-\infty}^{\infty} dy g(y)e^{-iky}\int_{-\infty}^{\infty} dz f(z)e^{-ikz} = \hat{f}(k)\hat{g}(k).$$

One can arrive at the same result in the case of a discrete Fourier transform:

$$\begin{aligned}\hat{f}(k)\hat{g}(k) &= \left(\sum_y e^{-iky} f(y)\right)\left(\sum_z e^{-ikz} g(z)\right) \\ &= \sum_x e^{-ikx}\left(\sum_y g(y) f(x-y)\right) \\ &= \widehat{f * g}(k)\end{aligned}$$

where all the terms with an exponent equal to z have been collected on the right hand side of the second line. We also used (in both derivations) the property $e^{-iky}e^{-ikz} = e^{-ik(y+z)}$ which is the key property here.

4 Exact Integral for the Position Distribution

Coming back to our initial problem, we introduce

$$\hat{P}_N(k) = \int_{-\infty}^{\infty} e^{-ikx} P_N(x) dx$$

which can be written as $\hat{P}_N(k) = \hat{p}_1(k)\hat{p}_2(k)\ldots\hat{p}_N(k)$. In the case of identical steps, this simplifies to $\hat{P}_N(k) = (\hat{p}_1(k))^n$. Using now the inverse transform, the exact solution for the position of the walker will be

$$P_N(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\Pi_{n=1}^N \hat{p}_n(k) dk.$$

Of course, this might not be easy to integrate, but so far everything done here has been exact. Notice that this can be generalized to higher dimensions simply by using the appropriate form of the Fourier transform. This also provides us a formula for the moments of the distribution $M_{N,l} = \int P_N(x)x^l dx$. Usually one will have to do an asymptotic analysis as $n \to \infty$ to progress further.

5 Additivity of Cumulants

We now return to the cumulants previously introduced. We consider an equivalent definition of the cumulants, which is more general. We define the **cumulants** of the transition PDF, $c_{n,l}$ as

$$\log \hat{p}_n(k) = \sum_{l=0}^{\infty} \frac{(-ik)^l}{l!} c_{n,l}$$

or equivalently

$$c_{n,l} = i^l \frac{d \log \hat{p}_n(k)}{dk^l}\bigg|_{k=0}.$$

The moments and cumulants are related as follows:

$$\hat{p}_n(k) = e^{\sum_{l=0}^{\infty} \frac{(-ik)^l}{l!} c_{n,l}} = \sum_{l=0}^{\infty} \frac{(ik)^l}{l!} m_{n,l}$$

Denote the cumulants of $P_N(x)$ by $C_{N,l}$ to find

$$\hat{P}_N(k) = \hat{p}_1(k)\hat{p}_2(k)\ldots\hat{p}_N(k) = e^{\sum_{l=0}^{\infty} \frac{(-ik)^l}{l!} \sum_{n=1}^{N} c_{n,l}}$$

which implies that $C_{N,l} = \sum_{n=1}^{N} c_{n,l}$, and thus shows that cumulants are additive for a sum of independent random variables. In the case of identical steps, $C_{N,l} = Nc_l$.

The first cumulants are easy to express in terms of moments, but the expressions get rapidly quite complicated.

$$\begin{aligned}
c_0 &= 0 \\
c_1 &= < x >= m_1 = \mu \\
c_2 &= < (x-\mu)^2 >= m_2 - m_1^2 = \sigma^2 \\
c_3 &= < (x-\mu)^3 >= m_3 - 3m_2m_1 + 2m_1^3 \\
c_4 &= < (x-\mu)^4 > -3\sigma^4 = m_4 - 4m_1m_3 + 6m_2m_1^2 - 3m_1^4 - 3(m_2 - m_1^2)^2
\end{aligned}$$

The **normalized cumulants** are simply defined by $\lambda_n = \frac{c_n}{\sigma^n}$ and the kurtosis is $\lambda_4 = \kappa$. Note that the Gaussian has $\hat{p}(k) = e^{-k^2\sigma^2/2}$ such that $c_2 = \sigma^2$ and all the higher cumulants are zero. In this sense, the cumulants $(n > 2)$ measure the departure from a Gaussian behaviour.

6 Example of a Gaussian Random Walk

Consider the example of the random walk in which the walker takes independent identical steps from a Gaussian distribution. Then we have

$$\begin{aligned}
p(x) &= \frac{e^{-x^2/(2\sigma^2)}}{\sqrt{2\pi\sigma}} \\
\hat{p}(k) &= e^{-\sigma^2k^2/2} \\
\hat{P}_N(k) &= (\hat{p}(k))^N = e^{-\sigma^2Nk^2/2}
\end{aligned}$$

This implies that $\sigma_N^2 = N\sigma^2$ is the variance after N steps. Inverting the previous result, we get

$$P_N(x) = \frac{e^{-x^2/(2\sigma_N^2)}}{\sqrt{2\pi\sigma_N^2}}$$

which means that the position distribution is Gaussian after every step. In other words, the Gaussian distribution is **stable under addition**. Note that that this agrees with our previous continuum approximation of the solution (to the diffusion equation) if we recall that $\zeta = x/\sqrt{\sigma_N} = x/\sqrt{2Dt}$, $t = N\tau$ and $D = \sigma^2/(2\tau)$.

7 Moments and Gram-Charlier Expansion

On another note, we recall that using continuum asymptotic methods previously we found the first corrections for a random flight with step distribution other than Gaussian to be

$$C(x,t) = \frac{e^{-\zeta^2/2}}{\sqrt{4\pi Dt}}\left(1 + \frac{\tau}{t}f_1(\zeta) + \ldots\right)$$

where we had assumed $\lambda_3 = 0$ and $\kappa \neq 0$. We found $f_1(\zeta)$ by ensuring that the moments smaller or equal to 2 were correct. More explicitly, this means

$$\begin{aligned}
<1> &= 1 \\
<x^2> &= 2Dt = \sigma^2 N \\
<x^4> &= \int_{-\infty}^{\infty} C(x,t)x^4 dx = \int_{-\infty}^{\infty} (2Dt)^2 \frac{e-\zeta^2/2}{\sqrt{2\pi}}\left(1 + \frac{\tau}{t}f_1(\zeta) + \ldots\right)\zeta^4 d\zeta.
\end{aligned}$$

From this we found $f_1(\zeta) = \frac{\kappa}{24}(3 - 6\zeta^2 + \zeta^4)$ to preserve the zeroth and second moments. We thus claimed that the fourth moment was also correct and we will now verify this. Let us denote by I_{2m} the even moments of the Gaussian and use the simple formula $I_{2m} = (2m-1)!! = (2m-1)(2m-3)\ldots 1$. The fourth moment of our solution is thus:

$$\begin{aligned}
<x^4> &= (2Dt)^2\left(I_4 + \frac{\tau}{t}\frac{\kappa}{24}(3I_4 - 6I_6 + I_8)\right) \\
&= (2Dt)^2\left(3 + \frac{\tau}{t}\frac{\kappa}{24}(3\cdot 3 - 6\cdot 15 + 15\cdot 7)\right)
\end{aligned}$$

$$<x^4>= (2Dt)^2\left(3 + \frac{\tau}{t}\kappa\right)$$

Using the definitions $t = N\tau$ and $D = \sigma^2/(2\tau)$ as before, this is equivalent to $M_4 = 3\sigma^4 N^2 + \sigma^4\kappa N$. But using the cumulants properties,

$$\begin{aligned}
C_4 &= M_4 - 3N^2\sigma^4 \\
C_4 &= Nc_4 = N\kappa\sigma^4
\end{aligned}$$

Therefore $M_4 = 3N^2\sigma^4 + N\kappa\sigma^4$, just as we had earlier claimed.