Lecture 6: Intermediate Asymptotics of Random Walks(Part II)

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[Continued from lecture 5...]

1 Calculation II : solving for the polynomial solution

The equation

$$(f_1'' - \zeta f_1' + 2f_1) = -\frac{\kappa}{12}(3 - 6\zeta^2 + \zeta^4) \qquad (1)$$

has a polynomial solution. Its homogeneous solution is of the form

$$f_1^h = A + B\zeta^2 + C\zeta^4$$

Putting into

$$(f_1'' - \zeta f_1' + 2f_1) = 0,$$

we get

$$f_1^h = 1 - \zeta^4$$

For particular solution f_1^p, it can be shown that

$$f_1^p = a + b\zeta^2 + \frac{\kappa}{24}\zeta^4,$$

where

$$a + b = -\frac{\kappa}{8}$$

Combining these solutions, we get

$$f_1^p = A + B\zeta^2 + \frac{\kappa}{24}\zeta^4, \quad A + B = -\frac{\kappa}{24} \qquad (2)$$

Now, how are the unknown coefficient A and B are determined? The asymptotics is not really arbitrary if you make the following assumption: each correction does not mess up the *MOMENTS* of the PDF of the walker's position at time t already correct at the previous level of approximation. By *"moment"*, we mean the integral of $x^n C(x,t)$. In the problem, the leading order solution has $M_0 = 1, M_1 = 0, M_2 = 2Dt$ (check it). Your correction should not screw up these moments. If so, it will turn out that the exact one, M_4 in this case, will be correct. With this reasonable (but at first mysterious) assumption, it can actually be shown that each new term in the expansion automatically gets the next moment correct. The Gaussian leading order solution gets moments of order 0, 1, and 2 exactly correct, and by requiring this the 4th moment will magically end up being correct too (since $M_3 = 0$). In the case of the homework, we made sure that the first two moments (0th = normalization, 1st = mean) were not perturbed.

There is also another way to do the problem using Fourier transforms which elucidates the connection with moments. This will discussed in next lecture.

[IDEA] *Don't mess up moments which are correct at the previous order of approximation.* The moments of normalized Gaussian distribution is easy to find.

$$M_{2m} := I_{2m} = \int \zeta^{2m}\Phi_0(\zeta)d\zeta$$

$$I_0 = 1, \quad I_2 = 1, \quad I_4 = 3, \quad I_6 = 15, \quad I_8 = 105, \quad I_{10} = 945,$$

$$\ldots, \quad I_{2m} = \frac{2m!}{m!2^m}$$

$$I_{2m} = (2m-1)!! := (2m-1)(2m-3)(2m-5)\cdots 3\cdot 1 \tag{3}$$

Using this,

$$M_0 = \int C_0(x,t)dx = 1 \Rightarrow \quad \int C_0(x,t)dx = 0$$

From M_0, we get $A + B = -\frac{\kappa}{8}$ which we already know of.

$$M_2 = \int x^2 C_0(x,t)dx = (2Dt)\int \zeta^2\Phi_0(\zeta)d\zeta = 2Dt$$

$$\Rightarrow \quad \int x^2 C_1(x,t)dx = 0$$

$$\Rightarrow \quad (2D)\int \frac{1}{2D}x^2(\frac{\tau}{t})\frac{\Phi_0(\zeta)f_1(\zeta)}{\sqrt{2Dt}}dx = 0$$

$$\Rightarrow \quad 2Dt\int \zeta^2 f_1(\zeta)\Phi_0(\zeta)d\zeta = 0$$

$$\Rightarrow \quad (2Dt)(AI_2 + BI_4 + \frac{\kappa}{24}I_6) = (2Dt)(A + 3B + 15\frac{\kappa}{24}) = 0$$

Where I_2, I_4 and I_6 are given in equation 3. Using $A + B = -\frac{\kappa}{8}$, we can find

$$A = \frac{\kappa}{8}, \quad B = -\frac{\kappa}{4}$$

.

Therefore,

$$f_1(\zeta) = \frac{\kappa}{4!}(3 - 6\zeta^2 + \zeta^4) \tag{4}$$

There might be other solutions for equation 1. For example, consider below as an another homogeneous solution for $f_1'' - \zeta f_1' + 2f_1 = 0$.

It looks like :

$$-2\zeta e^{\zeta^2/2} + 2\sqrt{2}\int_0^{\zeta/\sqrt{2}} e^{t^2}\, dt(\zeta^2 - 1)$$

The second term of this definitely blows up as $\zeta \to \inf$. It's OK for the solution to blow up (as does the simple $1 - \zeta^2$ solution), but it cannot blow up so fast that it beats the decay of the Gaussian factor.

Based only on intuition, it diverges faster than the Gaussian decays. Therefore, with this solution for $f_1(\zeta)$, the correction function $\Phi_1(\zeta) = \Phi_0(\zeta) f_1(\zeta)$ blows up (like -2ζ from the first term below), and thus the expansion cannot even be normalized. Hence, this homogeneous solution can be ruled out(thankfully!).

2 Role of the correction term

How does the correction term for 4th moment effects the distribution? As already told, 4th moment is related with the *tail* of the distribution. That is to say, it is a criterion of how much the tail of the distribution is different with that of the Gaussian distribution. Note that in the last lecture, we introduced the notion of *kurtosis* as a way of checking the shape of the tail. For example, we said that if $\kappa < 0$, then it will have thin tail. Now this can be explained from considering the equation 4.

See figure 1.

If $\kappa < 0$, the correction term has local minimum at $x = 0$, which means it suppresses the distribution at $(0, \frac{\kappa}{8})$. As x goes apart from the origin, it drastically decreases to negative value. Thus, when it is added to the 1st order term, the tails become thinner. On the contrary, if $\kappa > 0$, then now it has its local maximum at $(0, \frac{\kappa}{8})$ and its tail goes up one both sides. Thus, the resulting distribution graph should have fat tails.

Let's consider the case of binomial simple walk as an example. For unbiased simple walk, we know from last lecture that $\kappa = -2$ and thus it has thin tails.

It would be nice to compare graphically the leading order solution Φ_0 and the corrected solution above with the exact solution:

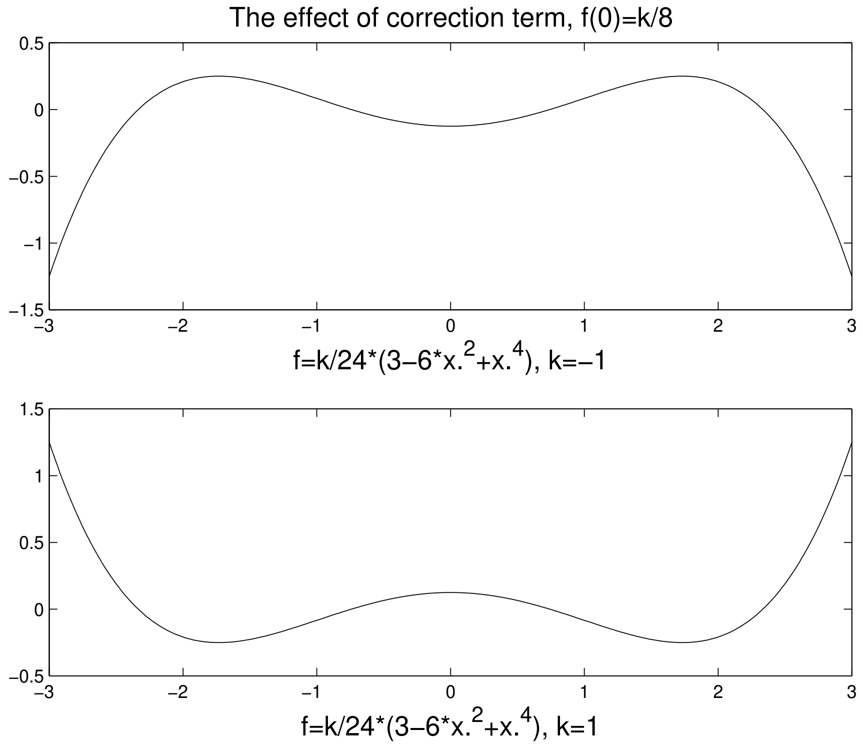

Figure 1: Correction term containing kurtosis : The case of $\kappa = -2$ and $\kappa = 2$ are compared.

$$p_N(m) = {}_N\mathrm{C_k p^k (1-p)^{n-k}}$$

where $k = \frac{n+m}{2}$ (which must be an integer, or that term is zero).

Here ${}_N\mathrm{C_k}$ is "N choose k", the binomial coefficient. The comparison might be easiest in the "x" variable.

Note that

0th order :

$$C_0(x,t) = \frac{1}{\sqrt{2Dt}}\Phi_0(\zeta) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}} \tag{5}$$

1st order correction :

$$C_0(x,t) + \frac{\tau}{t}C_1(x,t) = \frac{1}{\sqrt{2Dt}}\Phi_0(\zeta)(1 + \frac{\tau}{t}f_1(\zeta)), \quad f_1(\zeta) = \frac{\kappa}{4!}(3 - 6\zeta^2 + \zeta^4) \tag{6}$$

exact solution :

$$p_N(m) = {}_N\mathrm{C}_{\frac{\mathrm{N+m}}{2}} \frac{1}{2}^{\mathrm{(N+m)/2}} \frac{1}{2}^{\mathrm{(N-m)/2}} [1 + (-1)^{\mathrm{N+m}}]$$

$$= {}_N\mathrm{C}_{\frac{\mathrm{N+m}}{2}} 2^{-\mathrm{n}}[1 + (-1)^{\mathrm{N+m}}], \quad |\ \mathrm{m}\ | \leq \mathrm{n}$$

The relationship between the discrete case and corresponding continuous approximation is as follows(when the step size $a = 1$):

$$p_n(m) = 2C(ma, N\tau), \quad D = \frac{1}{2\tau}, \quad \zeta = \frac{x}{\sqrt{2Dt}} = \frac{m}{\sqrt{n}}, \quad \xi = \frac{\tau}{\sqrt{t}} = \frac{1}{\sqrt{N}} \tag{7}$$

where

$$x = m, \quad t = N\tau \tag{8}$$

Thus approximation to the simple walk on the integers in one dimension correct to first order is

$$p_n(m) \sim \Phi_0(z)\Big(1 - \frac{3 - 6z^2 + z^4}{12n} + O(\frac{1}{n^2})\Big)$$

where $z = \frac{m}{\sqrt{n}}$ and $\Phi_0(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$

Figure 2 shows the relationship among simple random walk, Gaussian distribution and its correction term. In the graph, binomial random walk is expressed as circles. Since binomial model itself has actual distribution of zero at every other node, other graphs were multiplied by a factor of 2 for appropriate comparison. Actually, we can see that the correction term itself becomes negligible as n increases. Also notice that already at $n = 4$, the error in the corrected expansion at first order is everywhere less than 5%.

We can express the graph using the ratio between the exact solution and continuous approximations(Gaussian and intermediate asympotics).

Figure 3 shows that the ratio between the intermediate asymptotics and exact solution is nearly 1, which indicates that the 1st order approximation is very accurate. As n becomes bigger, the Gaussian distribution works well also so that for $n = 16$, both provide 1 as its ratio within a certain domain.(Approximately for $x = (-10, 10)$). However, beyond that domain, the approximation fails and both the ratio blow. We call the domain where the approximation is valid ***"central region"***.

3 Width of the "central region"

The intermediate asymptotics approximation breaks down in the "tails" (large ζ) where the first correction is of the same size.

Since

$$C(x,t) \simeq \frac{e^{-\zeta^2/2}}{\sqrt{4\pi Dt}}\Big(1 + \frac{\tau}{t} f_1(\zeta) + \cdots\Big)$$

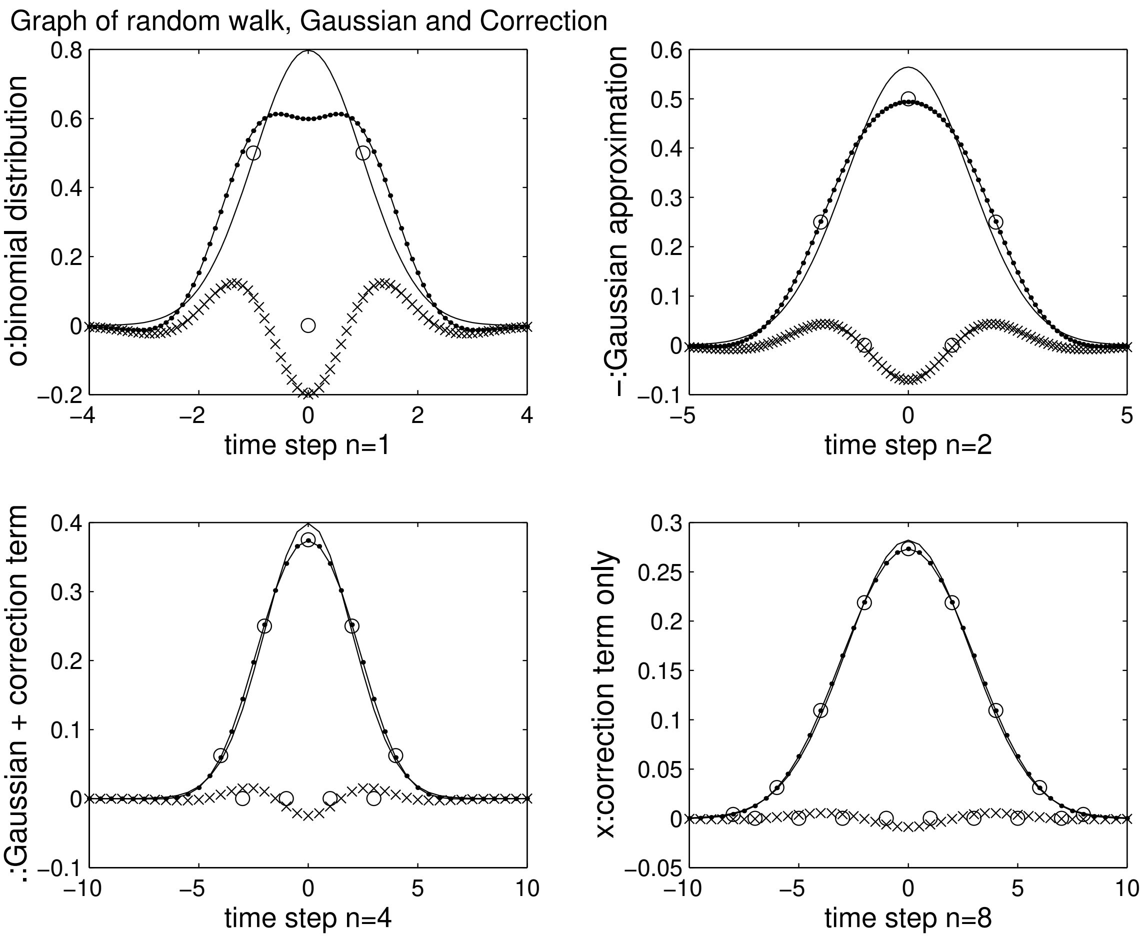

Figure 2: Comparison of intermediate asympotoics among simple random walk, Gaussian distribution and its correction term containing Kurtosis :binomial distribution(o), Gaussian(line), Gaussian+correction term(dot), Correction term only(x) for n=1, 2, 4, 8. Note that the correction term itself becomes negligible as n increases.

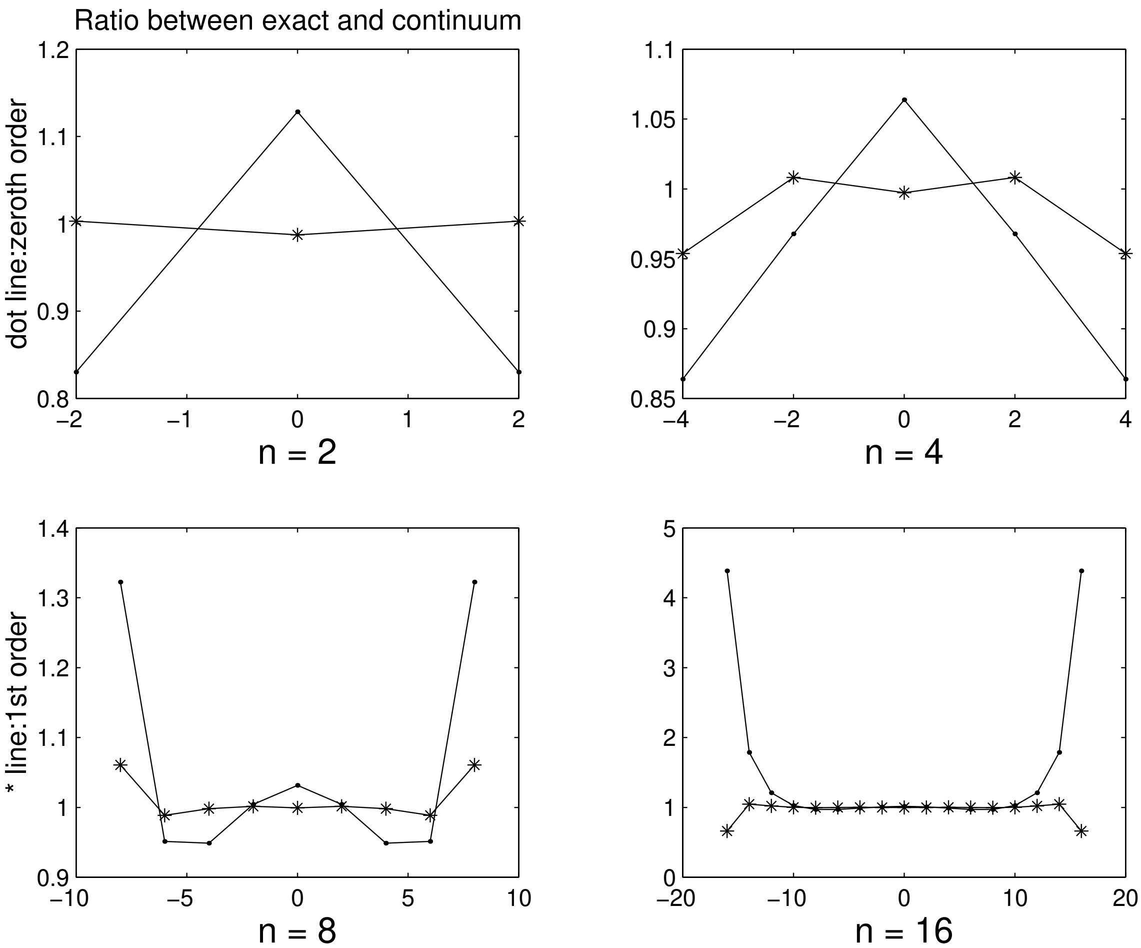

Figure 3: Ratio Comparison of intermediate asymptotics with the exact solution for a simple walk(Symmetric case, dimension=1): Gaussian is referred to *0th order* and Gaussian+correction term is referred to *1st order.*

as $\frac{\tau}{t} \to \infty$, ζ fixed, approximation fails(breaks down) when

$$\frac{\tau}{t} f_1(\zeta) \quad \simeq \quad 1 \quad \simeq \quad \frac{\tau}{t}\frac{\kappa}{24}\zeta$$

Solving for ζ, we find the *"the edge of central region ζ"*.

$$\zeta_{edge} \simeq \left(\frac{24t}{\kappa\tau}\right)^{1/4}$$

The relationship between ζ and x is again

$$\zeta = \frac{x}{\sqrt{2Dt}} = \frac{x}{\sqrt{2D\tau}}\sqrt{\frac{\tau}{t}}$$

Therefore,

$$x_{edge} \simeq \sqrt{2D\tau}\left(\frac{24t}{\kappa}\right)^{1/4}\left(\frac{t}{\tau}\right)^{3/4}$$

We can change this in case of discrete random walk on integers using equation 7 and 8.

$$m_{edge} \simeq \left(\frac{24}{\kappa}\right)^{1/4} n^{3/4} \tag{9}$$

This approximation explains the case $n = 16$ in figure 3. Applying equation 9 in this case, we get

$$12^{1/4} n^{3/4} \simeq 14.9$$

Thus, at $x \simeq x_{edge}$, the approximation is already not good. We can say that the *central region* in this case is

$$n^{3/4} \simeq 2^3 = 8 \text{ for } n = 8,$$

as is indicated in case of 1st order approximation(expressed as '*').

In case of Gaussian distribution, $\kappa = 0 \Rightarrow m_{edge} = \infty$ and thus x_{edge} is perfect.