Lecture 5: Intermediate Asymptotics of Random Walks(Part I)

Martin Z. Bazant

Department of Mathematics, MIT

22nd February, 2001(by Lee, Young Han)

1 Introduction

In this lecture, we will talk about two different kinds of intermediate asymptotics. The first one is to solve the diffusion equation with different initial conditions. Note that when the initial condition is dirac-delta function, we have the Green function as a solution. However, if the initial condition is given differently, for example, as Gaussian distribution, then we have width l in I.C. which creates another similarity variable.

If we start with such an initial distribution, then in the long time limit, the solution tends to spread out and the leading order behavior is given by the Gaussian distribution. It can be expressed as following:

$$C(x,t) = M_0 G(\zeta, t)$$

However, if we consider the correction terms, then similarity variables(such as $\xi = \frac{x}{\sqrt{2Dt}}$) appear.

$$M_0 G(\zeta, t)\Big(1 + f_1(\xi)\frac{l^2}{t} + f_2(\xi)\frac{l^4}{t^2} + ...)$$

This was dealt with in last class.

The second application is the discrete random walk. It can be said to be *'the Green function in discrete problem'*. If we only consider the leading order approximation, it will also be approximated by Gaussian distribution in the long run. However, if we consider the next order correction term, the result is different with the first case. Corrections are of order $\xi = \frac{\tau}{t}$ which occurs due to discreteness. Thus, it becomes different with the first case. Note that in this case, τ is fixed and $t \to \infty$.

However, these two are not really different. We can combine these two cases. The case can be regarded as bunch of random walkers start at one initial point and diffuse away. This is the same with the diffusion of delta function.(Up to the leading order.) It will be discussed later.

IN this lecture, we will focus on the case of discrete random walk. We will use diffusion equation and perturbation to solve correction term on $\xi = \frac{\tau}{t}$.

2 Formulation of discrete random walk

For a general random walk with iid(independent, identically distributed) random steps with transition probability $p_\tau(y)$:

The condition *independence* gives

$$C(x, t+\tau) = \int p_\tau(x, \acute{x}, t) C(\acute{x}, t) d\acute{x} = \int p_\tau(x - \acute{x}) C(\acute{x}, t) d\acute{x}$$

The later equality holds because it is iid and thus it depends on only $x - \acute{x}$ (where you are), not on t. This is essentially the Bachelier equation that we introduced before.

Now we consider the moments of the transition probability,

$$M_n(\tau) = \int x^n p_\tau(x) dx$$

This does depend on τ.

Let's assume two things in this lecture. First, we assume *zero mean*. We can always get this by considering the drift. In case of biased random walk, we can just shift it by the amount of drift by *change of variables*. Secondly, we assume the higher moments are all finite. That is,

$$M_1 = 0, \quad M_n < \infty$$

Then we have diffusion equation for this.

$$\frac{\partial C}{\partial t} - \mathcal{D}_2 \frac{\partial^2 C}{\partial x^2} = -\mathcal{D}_3 \frac{\partial^3 C}{\partial x^3} \tau^{1/2} + \left(\mathcal{D}_4 \frac{\partial^4 C}{\partial x^4} - \frac{1}{2} \frac{\partial^2 C}{\partial t^2} \right) \tau - \mathcal{D}_5 \frac{\partial^5 C}{\partial x^5} \tau^{3/2} + O(\tau^2) \tag{1}$$

where

$$D_N = \frac{M_n}{n! \tau} = O(\tau^{n/2-1})$$

as $\tau \to \infty$ and,

$$\mathcal{D}_n = D_n / \tau^{n/2-1}, n \geq 2$$

Note that the unit is

$$[\mathcal{D}_n] = [\mathcal{D}_2]^{n/2}$$

Now let's write the general solution by dimensional analysis. Using conservation of mass, we have

$$\boxed{C(x,t) = \frac{M_0}{(2Dt)^{d/2}}\Phi\Big(\frac{x}{\sqrt{2Dt}}, \frac{\tau}{t}, \frac{\mathcal{D}_3}{\mathcal{D}_2^{3/2}}, \frac{\mathcal{D}_4}{\mathcal{D}_2^2}, \ldots\Big)}$$

where

$$\zeta = \frac{x}{\sqrt{2Dt}}, \quad \xi = \frac{\tau}{t},$$

In case of the single random walker, we can set $M_0 = 1$.

When we say *intermediate limit*, it means that $\frac{\tau}{t} = \xi \to 0$ with ζ fixed. This can mean $\tau \to 0$ with t fixed or $t \to 0$ with τ fixed.

If you would like to take a long time limit, you can fix t and let τ go to 0. This is frequently used in finance when valuing option in continuous time limit.

3 Calculation

In this lecture, we will consider the case of symmetric walk. Then, all odd moments vanish and thus

$$\mathcal{D}_{2k+1} = 0$$

In the *intermediate limit*($\xi \to 0, \quad \zeta$ fixed), we assume that Φ has a regular Talyor expansion of the form

$$\Phi(\zeta,\xi) = \Phi_0(\zeta) + \xi\Phi_1(\zeta) + \xi^2\Phi_2(\zeta) + \ldots$$

Where $\Phi_1(\zeta) = \frac{\partial\Phi}{\partial\xi}(\zeta, 0), \ldots$
This assumption can be checked by substituting into the original equaion 1.

Note that if it were a biased walk, it would contain odd moment terms such as $\mathcal{D}_3$ and $\mathcal{D}_5$, and so on. $\mathcal{D}_3$ term has a coefficient of $\xi^{1/2}\Phi_{1/2}$. If $\mathcal{D}_5 \neq 0$, then $\xi^{3/2}\Phi_{3/2}$ will be contained in the equation.

This equation can be solved by regular perturbation. We equate the coefficients of ξ term according to its order.Be careful that $\xi = \frac{\tau}{t}$ and thus a variable of t. Since the equation contains partial differentiation with respect to t, ξ will also be differentiated. Likewise, $\zeta = \frac{x}{\sqrt{2Dt}}$ is a function of both x and t. Thus, it is affected by both $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$.

At $O(1)$:

$$\left(\frac{\partial}{\partial t}-\mathcal{D}_2\frac{\partial^2}{\partial x^2}\right)\frac{\Phi_0(\zeta)}{\sqrt{2Dt}}=0 \ \Leftrightarrow \Phi_0''(\zeta)+\zeta\Phi)0'(\zeta)+\Phi_0(\zeta)=0 \ \Leftrightarrow \Phi(\zeta)=\frac{e^{-\zeta^2/2}}{\sqrt{2\pi}}$$

This is the same as we did for 1st order approximation using Gaussian distribution. Now, at $O(\xi)$:

$$\left(\frac{\partial}{\partial t}-\mathcal{D}_2\frac{\partial^2}{\partial x^2}\right)\frac{\xi\Phi_1(\zeta)}{\sqrt{2Dt}}=t\xi\left(\mathcal{D}_4\frac{\partial^4}{\partial x^4}-\frac{1}{2}\frac{\partial^2}{\partial t^2}\right)\frac{\Phi_0(\zeta)}{\sqrt{2Dt}}$$

To solve this equation, we should do tedious calculations. Again, note that ξ is also a function of t and ζ is a function of t and x.

There are some formulas that are repeatedly used all through the calculation. They are given below:

1.
$$\frac{\partial\zeta}{\partial x}=\frac{1}{\sqrt{2Dt}},\quad \frac{\partial\zeta}{\partial t}=-\frac{D\zeta}{2Dt},$$
$$\frac{\partial\xi}{\partial x}=0,\quad \frac{\partial\xi}{\partial t}=-\frac{\tau}{t^2}$$

2.
$$\frac{\partial}{\partial t}(2Dt)^{-1/2}=-D(2Dt)^{-3/2},\quad \frac{\partial^2}{\partial t^2}(2Dt)^{-1/2}=-3D^2(2Dt)^{-5/2}$$

3.
$$\frac{\partial}{\partial x}\Phi=\Phi'(2Dt)^{-1/2},\quad \frac{\partial^2}{\partial x^2}\Phi=\Phi''(2Dt)^{-1},$$
$$\ldots,\quad \frac{\partial^n}{\partial x^n}\Phi=\Phi^{(n)}(2Dt)^{-n/2}$$

4.
$$\int e^{-ax^2}dx=\sqrt{\pi}a^{-1/2},\quad \int x^2e^{-ax^2}dx=\frac{1}{2}\sqrt{\pi}a^{-3/2},$$
$$\ldots,\quad \int x^{2n}e^{-ax^2}dx=\frac{1\cdot 3\cdots(2n-1)}{2^n}\sqrt{\pi}a^{-(2n+1)/2}$$

5.
$$\Phi_0=\frac{1}{\sqrt{2\pi}}e^{-\zeta^2/2},\quad \Phi_0'=-\zeta\Phi_0,$$
$$\Phi_0''=(\zeta^2-1)\Phi_0,\quad \Phi_0'''=(-\zeta^3+3\zeta)\Phi_0,$$
$$\Phi_0''''=(\zeta^4-6\zeta^2+3)\Phi_0,\quad \ldots$$

We calculate RHS first. RHS becomes

$$(-1)\frac{2D_2^2\Phi_0(\zeta)}{(2\mathcal{D}_2 t)^{5/2}}\left(\frac{\mathcal{D}_2^2-2\mathcal{D}_4}{4\mathcal{D}_2^2}\right)(3-6\zeta^2+\zeta^4)$$

Actually, the term $\left(\frac{\mathcal{D}_2^2-2\mathcal{D}_4}{4\mathcal{D}_2^2}\right)$ has a special meaning.

$$\left(\frac{\mathcal{D}_2^2-2\mathcal{D}_4}{4\mathcal{D}_2^2}\right)=\frac{1}{4}\left(1-\frac{2M_4/4!}{(M_2/2)^2}\right)=\frac{1}{4}\left(1-\frac{M_4}{3M_2^2}\right)=-\frac{1}{12}\kappa$$

where

$$\kappa=3\frac{2\mathcal{D}_4-\mathcal{D}_2^2}{\mathcal{D}_2^2}=\frac{M_4}{M_2^2}-3$$

This is named *kurtosis*. It measures how the tails of distribution differ from those of Gaussian distribution.

[Classification of probability distribution according to *kurtosis*]

1. $\kappa = 0 \Rightarrow$ [Gaussian distribution]

$$M_2=\int x^2\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx=1,\quad M_4=\int x^4\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx=3$$

$$\Rightarrow \kappa=\frac{M_4}{{M_2}^2}-3=0$$

2. $\kappa < 0 \Rightarrow$ The distribution decays faster than Gaussian distribution. The tails are not as wide as Guassian distribution

   (eg1.) [uniform distribution]

$$M_2=\int_0^1 x^2dx=\frac{1}{3},\quad M_4=\int_0^1 x^4dx=\frac{1}{5}$$

$$\Rightarrow \kappa=\frac{M_4}{{M_2}^2}-3=-\frac{6}{5}$$

   (eg2) [simple walk]

$$M_2 = 1^2\frac{1}{2} + (-1)^2\Big(1-\frac{1}{2}\Big) = 1, \quad M_4 = M_2 = 1^4\frac{1}{2} + (-1)^4\Big(1-\frac{1}{2}\Big) = 1$$

$$\Rightarrow \kappa = \frac{M_4}{{M_2}^2} - 3 = -2$$

3. $\kappa > 0 \Rightarrow$ The distribution decays slower than Gaussian distribution(fat tails)

   (eg) [leptokurtotic]

$$p(x) \propto \frac{1}{(1+x^2)^3}$$

Next, let's calculate LHS.
Recall that we have

$$\tau = \frac{\xi}{t}, \quad C_1 = \xi\Phi_1$$

and

$$C_1(\zeta,t) = \frac{1}{t}(C_0(\zeta,t)f_1(\zeta))$$

Using the following change of variable,

$$\underbrace{\Phi_1(\zeta)}_{unknown} = \underbrace{\Phi_0(\zeta)}_{known}\ \underbrace{f_1(\zeta)}_{unknown}$$

LHS becomes

$$\frac{2D_2^2\Phi_0(\zeta)}{(2\mathcal{D}_2 t)^{5/2}}(-f_1'' + \zeta f_1' - 2f_1)$$

Equating LHS=RHS, we get

$$(f_1'' - \zeta f_1' + 2f_1) = \Big(\frac{\mathcal{D}_2^2 - 2\mathcal{D}_4}{4\mathcal{D}_2^2}\Big)(3 - 6\zeta^2 + \zeta^4)$$

It can be shown that the solutions are polynomials.

[To be continued in lecture 6...]