Lecture 21: Continuous-Time Random Walks

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Until today, we considered discrete-time walks with a known waiting time. Now the waiting time is random. One example of this is an atom moving through a crystal. It has a waiting time, τ, between steps. The probability density function for the waiting time is $\psi(\tau) = \frac{1}{\bar{\tau}} e^{\frac{-\tau}{\bar{\tau}}}$, where $\bar{\tau}$ is the mean waiting time.

1 Random Waiting Times

This is the study of τ by itself. This can also be thought of as a renewal process with only births. It is seen this way in statistics, especially those involving populations. Since τ is greater than zero, it is convenient to use Laplace transforms. $\tilde{f}(s)$ is the Laplace transform of $f(x)$, and $\hat{f}(k)$ is the Fourier transform of $f(x)$. The Laplace transform always converges in a half-plane, Re$s > c'$. The definitions of the Laplace transform and its inverse are:

$$\tilde{\psi}(s) = \int_0^\infty e^{-st} \psi(t) dt$$

$$\psi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \tilde{\psi}(s) ds.$$

In the inverse transform the path of integration is the "Bromwich contour", which is parallel to the imaginary axis anywhere in the half-plane of convergence of the Laplace transform ($c > c'$). As with Fourier transforms, the mth moment is easily expressed in terms of derivatives of the Laplace transform: $\bar{\tau^m} = (\frac{-d}{ds})^n \tilde{\psi}(s)$ evaluated at $s = 0$. Expanding this gives the asymptotic expansion $\psi(s) = 1 - \bar{\tau} s + \frac{\bar{\tau}^s s^2}{2} - \ldots$ as s goes to zero.

Here is Laplace convolution:

$$\psi_2(t) = \int_0^t \psi(\tau) \psi(t - \tau) dt = \psi * \psi(t)$$

The Laplace convolution is the probability density function for the time for the nth step at time t. This is the same as the usual convolution since $\psi = 0$ for $t < 0$. So $\psi_n(t) = \psi^{*n}(t)$ holds, and the convolution theorem is $\tilde{\psi}_n(s) = \tilde{\psi}(s)^n$.

If a finite mean exists (call it $\bar{\tau}$), then $\bar{\psi}(s) \sim 1-\bar{\tau}s$ as s goes to 0 and $\tilde{\psi}_n(s) \sim (1-\bar{\tau}s)^n = (1-\frac{\bar{\tau}sn}{n})^n \sim e^{-\bar{\tau}ns}$ as n goes to infinity with $w = ns$ fixed (s goes to zero). This implies the Strong Law of Large Numbers, i.e. "$\psi_n(t) \sim \delta(t-n\bar{\tau})$" in the sense that the variable $t/n\bar{\tau}$ tends to 1 in probability. This means that a continuous time random walk is equivalent to a discrete time random walk whenever there is a finite mean waiting time, which is the usual case. As n goes to infinity, such a CTRW is the same as a discrete-time walk with $t_n = n\bar{\tau}$. This can also be seen by studying the convolutions:

1. $\bar{\tau} < \infty$ and $\sigma_\tau^2 < \infty$. This implies that $\psi_n(t)$ approaches a Gaussian centered at $n\bar{\tau}$ with a width of $\sqrt{n}\sigma_\tau$, according to the Central Limit Theorem.

2. $\bar{\tau} < \infty$ and $\sigma_\tau^2 = \infty$. In an important class of examples, we have $\tilde{\psi}_n(s) \sim 1-\bar{\tau}s+a|s|^\alpha$. In this case, $\psi_n(t)$ tends to an asymmetric Lévy distribution $l_{\alpha,1}\left((t-n\bar{\tau})/bn^{1/\alpha}\right)$ with $\beta = 1$ centered at $n\bar{\tau}$. However, although the variance is infinite, the width of the distribution $n^{1/\alpha}$ is still much smaller than the mean $n\bar{\tau}$, and hence the distribution of $t/n\bar{\tau}$ tends to 1 in probability, as long as $1 < \alpha \leq 2$.

1.1 Infinite Mean Waiting Time and One-Sided Stable Laws

This is when $\bar{\tau} = \infty$ or $\alpha \leq 1$. The general Lévy stability arguments from the previous lecture still hold: When appropriately rescale, $\psi_n(t)$ will tend toward a "one-sided" ($\beta = 1$) Lévy stable law $l_{\alpha,1}(t)$. These one sided stable laws can be represented using Laplace transforms as well:

$$\tilde{l}_{\alpha,1}(s) = e^{-s^\alpha}.$$

This clearly satisfies the stability equation,

$$\tilde{\psi}(n^{1/\alpha}s) = \tilde{\psi}(s)^n,$$

which is of the same form as the Fourier representation for symmetric laws ($\beta = 0$).

Now let us show that the simple Laplace representation of the stable laws is equivalent to the Fourier representation derived in the previous lecture.

$$l_{\alpha,1} = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{sx-s^\alpha} ds$$

Deform the path of the integral (the "Bromwich contour") so that it goes along the imaginary axis, which is justified because the integrand decays sufficiently fast at infinity, and no singularities are encountered. Use the change of variables $s = ik$ (i.e. rotate the plane by $\pi/2$) in the integral to get

$$l_{\alpha,1} = \frac{1}{2\pi}\int_{\infty}^{\infty} e^{ikx-(ik)^\alpha} dk$$

Note that $(ik)^\alpha$ is equal to $e^{\frac{i\pi\alpha}{2}}k^\alpha$ when k is greater than zero and $e^{\frac{-i\pi\alpha}{2}}|k|^\alpha$ when k is less than zero. This is equivalent to $|k|^\alpha(\cos\frac{\alpha\pi}{2} - \mathrm{sgn}(k)\sin\frac{\alpha\pi}{2}$. Therefore the Fourier transform of the one sided law is

$$\hat{l}_{\alpha,1}(k) = e^{a|k|^\alpha(1-i\text{sgn}(k)\tan\frac{\alpha\pi}{2})}$$

where $a = \cos\frac{\alpha\pi}{2}$, the same as obtained before ($\alpha < 1$, $\beta = 1$).

1.2 Smirnov Density

There is a one sided stable law which can be easily expressed in terms of elementary functions: $\alpha = \frac{1}{2}$. Then $\tilde{l}_{\alpha,1}(s) = e^{-bs^{1/2}}$. This has a branch cut along the negative real axis. Integrate this by moving the contour of integration so that it wraps around the branch cut. For $\tau > 0$, these extra contributions go to zero. Then the integral becomes

$$\frac{1}{2\pi i}\int_{c+i\infty}^{c-i\infty} e^{st-s^\alpha b}ds.$$

Then some algebra ensues:

$$\begin{aligned} l_{\frac{1}{2},1}(t) &= \frac{1}{2\pi i}\int_0^\infty e^{-xt}(-e^{-bi\sqrt{x}} + e^{bi\sqrt{x}})dx \\ &= \frac{1}{\pi}\int_0^\infty e^{-xt}\sin(b\sqrt{x})dx \\ &= \frac{1}{\pi}\int_0^\infty e^{-xt}\sum_{m=0}^\infty \frac{(b\sqrt{x})^{2m+1}}{(2m+1)!}(-1)^m dx \end{aligned}$$

In this case the series converges asymptotically for $\alpha > 1$.

$$\frac{1}{\pi}\sum_{m=0}^\infty \frac{(-1)^m b^{2m+1}}{(2m+1)!}\int_0^\infty e^{-xt}x^{m+\frac{1}{2}}dx$$

This integral is equal to $\frac{\Gamma(m+\frac{3}{2})}{x^{\frac{m+3}{2}}}$. Think about the gamma function: $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $\Gamma(\frac{5}{2}) = \frac{7}{2}\frac{\sqrt{\pi}}{2}$, $\Gamma(\frac{1}{2}) = \frac{5}{2}\frac{3}{2}\frac{\sqrt{\pi}}{2}$, $\Gamma(m+\frac{3}{2}) = \frac{(2m+1)!!\sqrt{\pi}}{2^{m+1}}$. Also, $\frac{(2m+1)!!}{(2m+1)!} = \frac{1}{eventerms} = \frac{1}{2m2(m-1)2(m-2)...2} = \frac{1}{m!2^m}$ Then

$$l_{\frac{1}{2},1}(t) = \frac{1}{\pi}\sum_{m=0}^\infty \frac{(-1)^m b^{2m+1}}{m!2^m}\frac{\sqrt{\pi}}{2^{m+1}}\frac{1}{t^{m\frac{3}{2}}} = \frac{b}{\sqrt{\pi}2t^{\frac{3}{2}}}\sum_{m=0}^\infty \frac{(-1)^m}{m!}(\frac{b^2}{4t})^m.$$

So the final answer for the Smirnov Density is

$$l_{1/2,1}(t) = (\frac{b^2}{4\pi t^3})^{1/2}e^{\frac{-b^2}{4t}}.$$

This is a one-sided stable law with $\alpha = \frac{1}{2}$. It has an essential singularity and $< t >= \infty$. It increases rapidly at the beginning, then falls off like $\frac{1}{t^{\frac{3}{2}}} = \frac{1}{t^{1+\alpha}}$.

2 The Random Number of Steps in a Given Time Interval

When the waiting time is very long, nothing will happen in a time t. So $\Psi(t)$ =Prob($\tau < t) = \int_t^\infty \psi(\tau)d\tau$. Note that $\tilde{\Psi}(s) = \frac{1}{s}(1 - \tilde{\psi}(s))$ by integration by parts. Let $\mathfrak{P}(n,t)$ be the probability that the walker has taken n steps before time t.

$$\mathfrak{P}(n,t) = \int_0^t \psi_n(\tau)\Phi(t-\tau)d\tau$$

because nothing happened after the nth step at time τ. Then convolution works as usual: $\psi_n * \Psi = \psi * \psi * \ldots * \psi * \Psi$. The Laplace transform of $\mathfrak{P}$ is

$$\tilde{\mathfrak{p}}(n,s) = \tilde{\psi}_n(s)\tilde{\Psi}_n(s) = \frac{\tilde{\psi}(s)^n(1-\tilde{\psi}(s))}{s}.$$

Note that $\bar{n}$, the expected number of steps taken before t, is

$$\bar{n}(s) = \sum_{n=0}^{\infty} n\tilde{\mathfrak{P}}(n,s) = \frac{\tilde{\psi}(s)}{s(1-\tilde{\psi}(s))}$$

when the geometric series is summed.

Example: Exponential waiting times and the Poisson Process

This occurs when $\psi(t) = \frac{1}{\tau}e^{\frac{-t}{\tau}} = \lambda e^{-\lambda t}$. $\lambda = \frac{1}{\tau}$ is the rate of stepping. Then $\tilde{\psi}(s) = \int_0^\infty \lambda e^{-(s+\lambda)\tau}d\tau = \frac{\lambda}{s+\lambda}$ and $\tilde{\mathfrak{P}}(n,s) = (\frac{\lambda}{s+\lambda})^n(1 - \frac{\lambda}{s+\lambda})\frac{1}{s} = \frac{1}{\lambda}(\frac{\lambda}{s+\lambda})^{n+1}$. Taking the inverse Laplace transform gives $\mathfrak{P}(n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{\lambda}(\frac{\lambda}{s+\lambda})^{n+1}ds$ This is analytic except for a simple pole at $s = -\lambda$, so it can be done using the Residue Theorem or using Cauchy's formula. Using Cauchy's formula, this is equal to $\lambda^n \frac{1}{2\pi i}\oint \frac{e^{st}ds}{(s+\lambda)^{n+1}} = \frac{\lambda^n}{n!}\frac{d^n}{ds^n}e^{st}|_{s=-\lambda}$. The last step comes from the Cauchy Derivative Formula. Then

$$\mathfrak{P}(n,t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

which is the Poisson distribution in n. So the steps occur according to a "Poisson process" with a mean of $\lambda t = \frac{t}{\tau} = \bar{n}$ and variance $\sigma_n^2(t) = \lambda t$.

3 Continuous-Time Random Walks

Continuous time random walks were introduced by E. Montroll and G. H. Weiss in 1965. The equation for such a walk is:

$$P(x,t) = \sum_{n=0}^{\infty} \mathfrak{P}(n,t)P_n(x)$$

P is the probability density function of being at x at time t. $\mathfrak{P}$ is the probability density function of taking n steps during the interval $(0,t)$. $P_n(x)$ is the probability density function for the position after n steps in the usual discrete-time walk. This is called a separable continuous-time random walk. This walk can be solved exactly using the Fourier transform for space:

$$\hat{P}(k,t) = \sum_{n=0}^{\infty} \mathfrak{P}(n,t)\hat{p}(k)^n$$

and the Laplace transform for time:

$$\tilde{\hat{P}}(k,s) = sum_{n=0}^{\infty}(\frac{1-\tilde{\psi}(s)}{s})\tilde{\psi}(s)^n\hat{p}(k)^n.$$

A useful trick is to sum the last part of the Laplace transform as a geometric series. The exact solution for the walk is:

$$\tilde{\hat{P}}(k,s) = (\frac{1-\tilde{(}s)}{s})\frac{1}{1-\tilde{\psi}(s)\hat{p}(k)}.$$

Unfortunately this is very hard to invert in most cases.

3.1 The Central Limit Theorem for CTRW

Convergence to the central limit theorem can be very different here. When the mean waiting time $\bar{\tau}$ and variance σ^2 are finite, the central limit theorem holds. This is $P(x,t) \sim \phi_0$, where $\phi_0(\zeta) = \frac{e^{\frac{-\zeta^2}{2}}}{2\pi}$. In this case $\zeta = \frac{x-\bar{x}(t)}{\sigma_{\Delta x}\sqrt{\frac{t}{\bar{\tau}}}}$. $\bar{x}(t)$ is $\bar{\Delta x}\frac{t}{\tau}$, a parameter to get rid of the drift.

3.1.1 First proof

As $\frac{t}{\tau} \to \infty, \bar{n} \to \infty$ because $n \to \frac{t}{\tau}$. Then $P(x,t) = \sum_n \mathfrak{P}(n,t)\phi_0(\frac{x}{\sigma\sqrt{n}})$. The idea here is that $\mathfrak{P}(n,t) \to \delta(n - \frac{t}{\bar{\tau}}$. This means that a continuous-time random walk with a finite mean waiting time can be renormalized into a discrete-time walk.

3.1.2 Second proof

First recall that $\hat{p}(k) \sim 1 - \frac{\sigma^2k^2}{2}$ and $\tilde{\psi}(s) \sim 1 - \bar{\tau}s$. Then go back to the Montroll-Weiss Equation:

$$\tilde{\hat{P}}(k,s) \sim \frac{\bar{\tau}}{\frac{\sigma^2k^2}{2} + \bar{\tau}s}$$

Doing the inverse Laplace transform gives

$$\hat{p}(k,t) \sim e^{\frac{-k^2\sigma^2}{2}\frac{t}{\bar{\tau}}}.$$

Then the inverse Fourier transform shows that this is indeed a Gaussian, $P(x,t) \sim \phi_0\frac{x}{\sigma\sqrt{\frac{t}{\bar{\tau}}}}$.

3.2 Exact solution of the Poisson-Bernoulli CTRW

To set up this walk, define $\psi(t) = \lambda e^{-\lambda t}$, where $\lambda = \frac{1}{\tau}$ is the stepping rate. Then $\tilde{\psi}(s)\frac{\lambda}{s+\lambda}$. Also say $\hat{p}(k) = \cos(ak)$ where k is between $-\pi$ and π. Then

$$\tilde{\hat{P}}(k,s) = (\frac{1-\tilde{\psi}(s)}{s})(\frac{1}{1-\tilde{\psi}(s)\hat{p}(k)})$$

$$= \frac{1}{s+\lambda}\frac{1}{1-\frac{\lambda}{s+\lambda}\cos(ak)} = \frac{1}{s+\lambda(1-\cos(ak))}.$$

Take the inverse Laplace transform to discover

$$\hat{p}(k,t) = \frac{1}{2\pi i}\int \frac{e^{st}ds}{s+\lambda(1-\cos(ak))} = e^{-\lambda t(1-\cos(ak))}.$$

Then take the inverse Fourier transform to discover

$$P(ma,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikma-\lambda t(1-\cos(ak))}dk.$$

So the final answer is

$$P(mt) = \frac{e^{-\lambda t}}{a}I_m(\lambda t),$$

where $I_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(m\theta)e^{-x\cos\theta}d\theta$ is a modified Bessel function. Look at the central region as $\lambda t \to \infty$ (when t is much larger than $\bar{\tau}$):

$$e^{-\lambda t(1-\cos(ak))} \sim e^{\frac{-\lambda t a^2 k^2}{2}}$$

is a Gaussian with width $a\sqrt{\lambda t}$!