Lecture 16: Saddle-Point Asymptotics for Random Walks

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Previous lectures have covered methods to derive asymptotic approximations for the probability distributions of different random walks. These methods, including Kramers-Moyal expansions and Fourier transforms, gave approximations that were valid in the central region of the distribution, but not in the tails. In this lecture, we study the method of Steepest Descent, which gives an asymptotic approximation that is globally valid. We begin with an overview of Laplace integrals, since they figure prominently in the method.

1 Laplace Integrals

For $\psi(x)$ real and analytic, the Laplace integral of ψ over the interval $[a, b]$ is given by

$$\int_a^b e^{-n\psi(x)} dx \tag{1}$$

where n is a positive integer. The integrand of (1) will take on its largest values when ψ achieves its minima, assuming that it has at least one such minimum on the interior of $[a, b]$. An example ψ and the corresponding Laplace integrand are shown in Figs. 1 and 2.

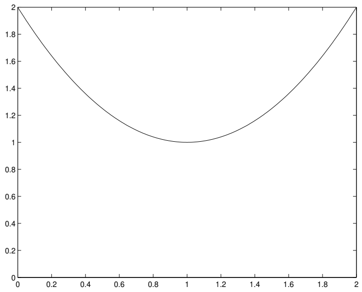

Figure 1: $\psi(x) = (x-1)^2 + 1$.

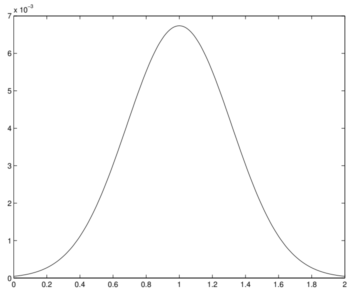

Figure 2: The Laplace integrand, $e^{-n\psi(x)}$, for the ψ of Fig. 1 and $n = 5$. As n increases, the width of the plot will decrease.

The approximation to the Laplace integral (1) is obtained in 2 stages. First, the function ψ is replaced by the first 3 terms of the Taylor expansion about its minimum point x_0,

$$\psi(x) \approx \psi(x_0) + \psi'(x_0)(x - x_0) + \frac{1}{2}\psi''(x_0)(x - x_0)^2. \tag{2}$$

Since x_0 is a minimum for ψ, then $\psi'(x_0) = 0$, and $\psi''(x_0) > 0$. (Note that in general, we may have $\psi''(x_0) = 0$, in which case we need to include more terms in the approximation (2). For this situation, the steps below need to be re-considered, especially the integral (4).) We next use the approximate relation (2), and change the limits of the Laplace integral to cover only the epsilon interval about x_0 :

$$\int_a^b e^{-n\psi(x)}dx \approx e^{-n\psi(x_0)} \int_{x_0-\epsilon}^{x_0+\epsilon} e^{-n\frac{\psi''(x_0)}{2}(x-x_0)^2}dx \tag{3}$$

Doing so introduces only exponentially small errors into the approximation for the integral.

The second stage of the approximation solution to the Laplace integral is perhaps the more surprising one. Here we allow ϵ from the right-hand side of (3) to approach ∞, and we find that the approximation to the original integral (1) still only involves exponentially small errors:

$$\int_a^b e^{-n\psi(x)}dx \approx e^{-n\psi(x_0)} \int_{-\infty}^{\infty} e^{-n\frac{\psi''(x_0)}{2}(x-x_0)^2}dx \tag{4}$$

The right-hand side of (4) is especially useful, since it is within a multiplicative constant of a Normal PDF, with mean x_0 and variance $\frac{1}{n\psi''(x_0)}$. Thus integrating out the rhs of (4) gives

$$\int_a^b e^{-n\psi(x)}dx \approx e^{-n\psi(x_0)}\sqrt{\frac{2\pi}{n\psi''(x_0)}} \tag{5}$$

Eq. (5) gives the approximate solution for the Laplace integral, if $\psi(x)$ is real. If instead ψ is complex, then we note that

$$|e^{-n\psi}| = e^{-nRe\psi},$$

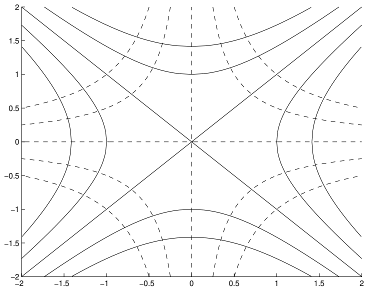

Figure 3: Plots of constant real and imaginary parts for $f(z) = z^2 = (x^2 - y^2) + 2ixy$. Solid lines are for constant $Re\ f$, and dashed are for constant $Im\ f$.

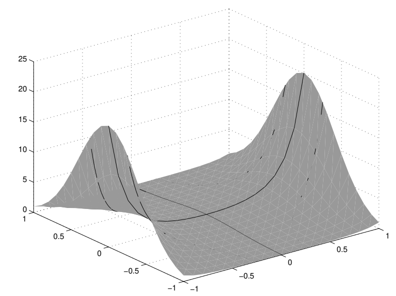

Figure 4: The surface $|e^{-nf}|$ for $f(z) = z^2$ and $n = 3$. Some lines of constant phase are superimposed.

which says that the size of the integrand is determined by the real part of ψ. In particular, if we allow ψ to have a complex argument, and take a contour integral of ψ through the complex plane along a path such that $Im\ \psi$ is constant, then

$$\int_C e^{-n\psi(z)} dz = e^{-nIm\psi} \int_C e^{-nRe\psi} dz \tag{6}$$

where the integral on the right-hand side is essentially a Laplace integral.

Next quoting a result from complex analysis, we note that since $e^{-n\psi}$ is analytic and $e^{-n\psi} \neq 0$ always, then the function $|e^{-n\psi}|$ cannot attain a max or min on the interior of any given domain, it can only attain a "saddle point."

We also note that, since analytic complex functions produce conformal maps of the complex plane, then curves of constant phase cannot intersect, except at points where $\psi'(z) = 0$.

These points are illustrated in Figs. (3) and (4) for the example function $f(z) = z^2$. Fig. (3) shows curves of constant $Re\ f$ and $Im\ f$, and Fig. (4) shows the surface $|e^{-nf}|$.

Thus the idea for solving Laplace integrals for complex ψ is the following: We take the original integral (1), deform the path of integration in the complex plane in such a way, so that the path of integration passes through the lowest saddle point of the function $|e^{-n\psi}|$ along a path of steepest descent. Then, the dominant contribution to the integral as $n \to \infty$ will come from the neighborhood of the saddle point. There may also be important contributions to the integral from the endpoints, if they correspond to values of $|e^{-n\psi}|$ larger than the value at the saddle point.

2 Bernoulli ("Simple") Random Walk

In this section we use the method of Steepest Descent to derive a globally valid asymptotic approximation for the probability distribution of the Bernoulli random walk, also referred to in previous lectures as the "simple" random walk. The example is given for iid steps and the $d = 1$, or 1-dimensional case.

The transition probability distribution for the Bernoulli random walk is given by

$$p(m) = \frac{1}{2}(\delta_{m,1} + \delta_{m,-1}) \tag{7}$$

where δ is the Kronecker delta, and $m \in Z$, since the problem is inherently discrete. This discreteness also dictates the use of the Discrete Fourier Transform, rather than continuous. (Note that by convention, the exponential term is taken as e^{imk} with the DFT, rather than using $-i$ as in the continuous case.) Adapting a computation from a previous lecture for the transform of the Bernoulli transition probability gives

$$\hat{p}(k) = \sum_{m=-\infty}^{\infty} e^{imk} p(m) = \cos(k) \tag{8}$$

Since the steps are iid, the transform of the probability distribution after n steps is

$$\hat{P}_n(k) = \cos^n(k), \tag{9}$$

The distribution itself is given by the inverse transform. Note that the interval of integration is $[-\pi, \pi]$, since we are working with the DFT here.

$$P_n(m) = \int_{-\pi}^{\pi} e^{-imk} \cos^n(k) \frac{dk}{2\pi} \tag{10}$$

Manipulating the integral in (10) gives:

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{-imk} \cos^n(k) \frac{dk}{2\pi} &= \left(\int_{-\pi}^{-\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} \right) e^{-imk} \cos^n(k) \frac{dk}{2\pi} \\
&= \left(\int_{\pi}^{\frac{3\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} \right) e^{-imk} \cos^n(k) \frac{dk}{2\pi} \\
&= \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right) e^{-imk} \cos^n(k) \frac{dk}{2\pi} \qquad (11)
\end{aligned}$$

where the second step follows due to the 2π-periodicity of the integrand. Next making the change of variables $k = k' + \pi$ in the second integral gives

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-im(k'+\pi)} \cos^n(k'+\pi)\frac{dk'}{2\pi}$$

$$= e^{-im\pi}\cos^n(\pi)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-imk'} \cos^n(k')\frac{dk'}{2\pi}$$

$$= (-1)^{n+m}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-imk'} \cos^n(k')\frac{dk'}{2\pi} \tag{12}$$

where the trig identity $\cos(u+v) = \cos\ u\ \cos\ v - sin\ u\ sin\ v$ was used for the second step, and the *sin* terms were dropped due to the fact that $sin\ \pi = 0$. Substituting (12) into (11) leads to the relation

$$P_n(m) = \left[1 + (-1)^{n+m}\right]\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-imk} \cos^n(k)\frac{dk}{2\pi} \tag{13}$$

which may also be written as

$$P_n(m) = \frac{\left[1 + (-1)^{n+m}\right]}{2\pi} I(n,m) \tag{14}$$

for $I(n,m)$ the Laplace integral given by

$$I(n,m) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-n\psi(k,\xi)} dk \tag{15}$$

$$\psi(k,\xi) = i\xi k - \log\ \cos\ k$$

$$\xi = \frac{m}{n}$$

The function ψ is called the "Cramer" function. It holds all the information about the random walk. Previously, we expanded ψ about $k = 0$, which gave information about the central region of the distribution. Here we instead use Laplace integrals, which leads to an approximation valid in the central region and in the tails.

Specifically, the procedure to study the long-time behavior of $P_n(m)$ will be to allow $n \to \infty$ in such a way that ξ remains constant. This will yield an approximate solution $P_n(n\xi)$ as a function of ξ, and the approximation will be valid for all allowed values of ξ.

The first k-derivative of ψ is

$$\frac{\partial\psi}{\partial k} = i\xi + \tan\ k \tag{16}$$

which implies that the critical points k_s solve the equation

$$\tan\ k_s = -i\xi \tag{17}$$

We may solve for k_s by writing out the *sin* and cos components of tan k in terms of exponentials, and making the substitution $\omega = e^{ik_s}$. At the first step, this leads to

$$\left(\frac{\omega - \frac{1}{\omega}}{2i}\right)\left(\frac{\omega + \frac{1}{\omega}}{2}\right)^{-1} = -i\xi \tag{18}$$

which can be solved for ω to obtain

$$\omega = \sqrt{\frac{1+\xi}{1-\xi}} \tag{19}$$

Substituting back for k_s yields

$$k_s = \frac{i}{2}\log\left(\frac{1-\xi}{1+\xi}\right) \tag{20}$$

Here we note that $-1 \le \xi \le 1$, by the definition of ξ with (15). $|\xi| = 1$ represents the limiting case in which all the steps are taken in the same direction; the steps above must be re-considered in this event.

Next we study the second k-derivative for ψ at k_s :

$$\frac{\partial^2\psi}{\partial k^2}(k_s,\xi) = sec^2(k_s)$$

$$= 1 + \tan^2(k_s)$$

$$\frac{\partial^2\psi}{\partial k^2}(k_s,\xi) = 1 - \xi^2 \tag{21}$$

which says that $\frac{\partial^2\psi}{\partial k^2}(k_s,\xi) > 0$ (except, as noted before, in the limiting case $|\xi| = 1$.) Hence, the conditions for approximating $I(n,m)$ with the methods from Section (1) are justified, and we have

$$I(n,m) \approx \sqrt{\frac{2\pi}{n\psi''(k_s)}}e^{-n\psi(k_s)} \tag{22}$$

Writing out the expression for $\psi(k_s)$ gives

$$\psi(k_s) = i\xi k_s - \log\ \cos\ k_s$$

$$= \frac{i^2\xi}{2}\ \log\left(\frac{1-\xi}{1+\xi}\right) - \log\sqrt{\frac{1}{1-\xi^2}}$$

$$= \frac{\xi}{2}\ \log\left(\frac{1+\xi}{1-\xi}\right) + \frac{1}{2}\ \log(1-\xi^2) \tag{23}$$

where the second step follows from the trig identity $sec^2\ z = 1 + \tan^2\ z$, along with Eq. (17). Thus Eq. (23), when coupled with Eqs. (14) and (22), yields the globally valid asymptotic approximation

$$P_n(n\xi) \approx \frac{1 + (-1)^{n+m}}{\sqrt{2\pi n(1 - \xi^2)}} e^{-n\psi_s(\xi)} \tag{24}$$

where $\psi_s(\xi) = \psi(k_s)$.

Next studying the behavior of $P_n(m)$, we define

$$\zeta \equiv \frac{m}{\sqrt{n}} \tag{25}$$

Then noting $\xi = \frac{\zeta}{\sqrt{n}}$, we let $n \to \infty$, in such a way that ζ stays fixed. Then $\xi \to 0$, and using the leading-order approximation $\log(1 + \epsilon) \approx \epsilon$, it is possible to show

$$\psi(\xi) \sim \frac{\xi^2}{2} \tag{26}$$

for small ξ, and we arrive at the approximation

$$P_n(\sqrt{n}\zeta) \sim \frac{1 + (-1)^{n+m}}{\sqrt{2\pi n}} e^{\frac{-\zeta^2}{2}} \tag{27}$$

which expresses the Central Limit Theorem in the central region.