Lecture 14: Exact Solutions for Correlated Walks

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Last time we learned about correlation functions. Today there is no general solution, some examples will be done.

1 Constructing correlated walks from uncorrelated ones

Given Δx_m independent random displacements, construct a correlated version, $\delta\tilde{x}_m$ such that

$$< \Delta x_m \Delta x_{m+m'} >_p = \begin{array}{cc} \sigma_m^2 & m' = 0 \\ \rho & m' = 1 \\ 0 & else \end{array} \tag{1}$$

where m' is the number of steps in the past being compared with the present step and the p stands for partial correlation - the actual path of the walker is not being considered. This assumes that σ_m is the given variance with a mean of 0. The walk is not correlated to steps that are very far in the past. To construct this sequence, do this:

Put $\Delta\tilde{x}_1 = \Delta x_1$ and $\frac{\Delta\tilde{x}_m}{\sigma_m} = \rho\frac{\Delta\tilde{x}_{m-1}}{\sigma_{m-1}} + \bar{\rho}\frac{\Delta x_m}{\sigma_m}$, where $\bar{\rho} = \sqrt{1-\rho^2}$. Here the new step is the previous step plus a new random part. Now check that this works:

$$\frac{< \Delta\tilde{x}_m^2 >}{\sigma_m^2} = \rho^2\frac{< \Delta x_{m-1}^2 >}{\sigma_{m-1}^2} + 2\rho\bar{\rho} < \Delta\tilde{x}_{m-1}\Delta x_m + \bar{\rho}^2\frac{< \Delta x_m^2 >}{\sigma_m^2} \tag{2}$$

Here $\frac{<\Delta x_{m-1}^2>}{\sigma_{m-1}^2} = 1$, $\frac{<\Delta x_m^2>}{\sigma_m^2} = 1$, and $< \Delta\tilde{x}_{m-1}\Delta x_m >= 0$ by construction, since $\Delta\tilde{x}$ and Δx are uncorrelated. Then this gives

$$\frac{< \Delta\tilde{x}_m^2 >}{\sigma_m^2} = \rho^2 + \bar{\rho}^2 = 1 \tag{3}$$

which was expected. Now, does the construct have partial correlation as claimed?

$$\frac{< \Delta\tilde{x}_m\Delta\tilde{x}_{m-1} >}{\sigma_m\sigma_{m-1}} = \rho\frac{< \Delta\tilde{x}_{m-1}^2 >}{\sigma_{m-1}^2} + \bar{\rho}\frac{\Delta x_m\Delta\tilde{x}_{m-1} >}{\sigma_m\sigma_{m-1}} \tag{4}$$

Remember that it is necessary to divide the left hand term by $\sigma_m \sigma_{m-1}$ to get the correlation coefficient. Then, on the right hand side, the first term goes to ρ and the second term is 0, so the answer is ρ, as claimed.

$$\frac{\tilde{\Delta x}_m}{\sigma_m} = \rho \frac{\tilde{\Delta x}_{m-1}}{\sigma_{m-1}} + \bar{\rho} \frac{\Delta x_m}{\sigma_m} \tag{5}$$

Figure 1 shows how ρ and $\bar{\rho}$ symbolize the correlation.

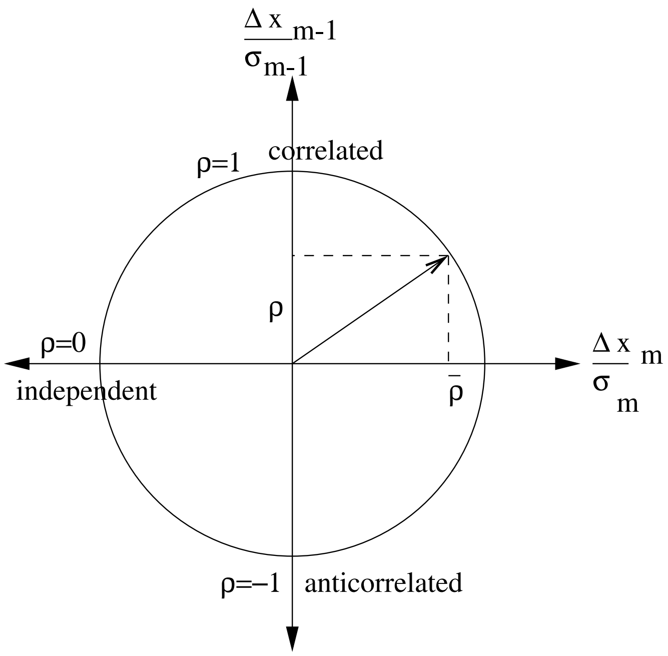

Figure 1: A circle showing correlation.

In the correlated Pearson walk, the correlation changes the step size. A new step is ρ times the old step plus a new random part that is multiplied by $\bar{\rho}$. This goes off the lattice. To keep the correlated walk on a lattice, higher moments are required. Each correlated step can be decomposed into a sum of independent steps:

$$\begin{aligned}
\tilde{\Delta x}_1 &= \Delta x_1 \\
\tilde{\Delta x}_2 &= \rho \Delta x_1 + \bar{\rho} \Delta x_2 \\
\tilde{\Delta x}_3 &= \rho^2 \Delta x_1 + \bar{\rho}\rho \Delta x_2 + \bar{\rho} \Delta x_3 \\
&\vdots \\
\tilde{\Delta x}_m &= \rho^{m-1} \Delta x_1 + \bar{\rho}\rho^{m-2} \Delta x_2 + \ldots + \rho\bar{\rho} \Delta x_{m-1} + \bar{\rho} \Delta x_m
\end{aligned} \tag{6}$$

Now look at the position of the walker after n steps, $\tilde{X}_n$.

$$\tilde{X}_N = \sum_{m=1}^{n} \tilde{\Delta}x_n = \begin{array}{c} \Delta x_1(1+\rho+\rho^2+\ldots+\rho^{n-1})+ \\ \Delta x_2\bar{\rho}(1+\rho+\rho^2+\ldots+\rho^{n-2})+ \\ \Delta x_3\bar{\rho}(1+\rho+\rho^2+\ldots+\rho^{n-3})+ \\ \vdots \\ \Delta x_n\bar{\rho} \end{array} \tag{7}$$

$$= \frac{(1-\rho^n)}{1-\rho}\Delta x_1 + \bar{\rho}\frac{1-\rho^{n-1}}{1-\rho}\Delta x_2 + \bar{\rho}\frac{1-\rho^{n-2}}{1-\rho}\Delta x_3 + \ldots + \bar{\rho}\Delta x_n = \sum_{m=1}^{n} a_m \Delta x_m \tag{8}$$

This is a correlated walk, just like it was before. Here the Δx_m are independent. The shape of the distribution is decaying: $\rho^k \to 0$, so everything goes to $\frac{1}{1-\rho}$. The correlations are exponentially decaying. Now to find the exact solution, as was done in a previous lecture:

$$\hat{P}_n(k) = \prod_{m=1}^{n} \hat{p}_m(a_m k) \tag{9}$$

Here $\hat{p}_m(k)$ is the structure function for Δx_m. This gives the Central Limit Theorem and a normal distribution, just like it did before, if $\frac{1}{n}\sum_{m=1}^{n} a_m^2 = \bar{\sigma}^2$ is the average variance. Then

$$< \tilde{X}_n^2 >= \sum_{m=1}^{n} a_m^2 \tag{10}$$

Now, sum the squares of all the constant terms of $\tilde{X}_n$:

$$\sum_{m=1}^{n} a_m^2 = (\frac{1-\rho^n}{1-\rho})^2 + (\frac{\bar{\rho}}{1-\rho})^2((1-\rho^{n-1})^2 + (1-\rho^{n-2})^2 + \ldots + (1-\rho)^2) + \bar{\rho}^2 \tag{11}$$

Neglecting the first and last terms and the large sum of $-\rho^n$s, and adding up all the 1s to get n, this gives:

$$\sum_{m=1}^{n} a_m^2 \sim n(\frac{\bar{\rho}}{1-\rho})^2 = n(\frac{1-\rho^2}{(1-\rho)^2}) = n\frac{(1+\rho)}{1-\rho}. \tag{12}$$

This is the same diffusion constant as before, $n\frac{(1+\rho)}{1-\rho}$!

2 The Persistent Walk

A persistent walker has a probability α of continuing in the same direction as the previous step. All other directions are equally likely, of probability $\frac{1-\alpha}{3}$ for the two-dimensional case. Then the correlation is given by:

$$< \Delta x_m \Delta x_{m-m'} >= \begin{array}{cc} 1 & m' = 0 \\ \rho & m' = \pm 1 \\ 0 & else \end{array} \tag{13}$$

where $\rho = 2\alpha - 1$. The differences between this and the Bernoulli walk are at higher order - look at the moments to see this. This is not the same as previous walks. Here the persistent walk will be analyzed in one dimension.

First set up $P_n(m) = A_n(m) + B_n(m)$, decomposing the position into two probabilities, A that the walker comes from the left and B that the walker comes from the right. The probability that the walker is at position m after n steps is $\frac{1}{2}Prob(X_n = m|X_{n-1} = m - 1) + \frac{1}{2}Prob(X_n = m|X_{n-1} = m + 1$, where the first term is the probability that the walker came from the left averaged over the whole path, and the second term is the same thing for a walker coming from the right. Sine the only information used is the probability that during the previous step the walker was at a neighboring position, this is based on the average of all previous steps.

Define $A_n(m)$ as the probability of being at m after n steps if the walker came from the left, and $B_n(m)$ as the same thing for a walker coming from the right. Let α be the probability of the walker going in the same direction as the previous step, and β the probability of changing direction. Then $\beta = 1 - \alpha$. Figure 2 shows how this can be expressed diagrammatically.

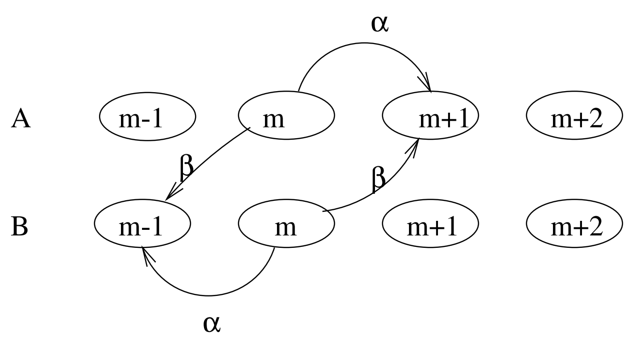

Figure 2: A diagram of the random walk. Changing direction is like changing levels on the diagram, so β signals a change of level. Continuing on in the same direction is like staying on the same level, so α is used.

This can be expressed as a state vector using A and B.

$$\begin{aligned} A_{n+1}(m) &= \alpha A_n(m-1) + \beta B_n(m-1) \\ B_{n+1}(m) &= \beta A_n(m+1) + \alpha B_n(m+1) \end{aligned} \tag{14}$$

This is called a Markov chain because it only uses knowledge from the previous timestep and no earlier timesteps. $A_n + 1$ is the probability that the walker came from the left. Then the first term, with α, is the probability that the walker came from the same level in the diagram, while the second term, with β is the probability that the walker will backtrack or change levels within the diagram. The state vector is $\begin{pmatrix} A_n(m) \\ B_n(m) \end{pmatrix}$. There are two ways to solve this problem now.

1. Solve it exactly, using the discrete Fourier transform.

2. Take the continuum limit, resulting in a partial differential equation.

The second method will be dealt with today and on problem set 2.

The goal of finding the continuum limit is to find $p(m\sigma, n\tau) = A_n(m) + B_n(m)$, where σ is the lattice spacing ($x = m\sigma$) and τ is the timestep ($t = n\tau$). To find p it is necessary to define $q(m\sigma, n\tau) = A_n(m) - B_n(m)$, $a(m\sigma, n\tau) = A_n(m)$, and $b(m\sigma, n\tau) = B_n(m)$ as well. Now derive a partial differential equation for p and q.

First, expand the Markov Chain equations around $x = m\sigma$, $t = n\tau$.

$$\begin{array}{c} a + \tau a_t + \frac{\tau^2}{2}a_{tt} + \ldots = \alpha(a - \sigma a_x + \frac{\sigma^2}{2}a_x x - \ldots) + \beta(b - \sigma b_x + \frac{\sigma^2}{2}b_{xx} - \ldots) \\ b + \tau b_t + \frac{\tau^2}{2}b_{tt} + \ldots = \beta(a + \sigma a_x + \frac{\sigma^2}{2}a_x x + \ldots) + \alpha(b + \sigma b_x + \frac{\sigma^2}{2}b_{xx} + \ldots) \end{array} \tag{15}$$

Add these two equations, keeping in mind that $\alpha + \beta = 1$ by construction since they are probabilities and that $a + b = p$ and $a - b = q$.

$$p + \tau p_t + \frac{\tau^2}{2}p_{tt} + \ldots = (\alpha + \beta)p + (\beta - \alpha)\sigma q_x + \frac{(\alpha - \beta)\sigma^2}{2}p_{xx} - \ldots \tag{16}$$

Note that the p terms cancel out since $\alpha + \beta = 1$. Dividing by τ makes the third term on the left have $\frac{\sigma}{\tau}$, so the second term on the left and the third term on the right are normal diffusion. The only remaining term on the right, the one with q_x, is due to the correlation.

Subtracting the two equations in 15 gives similarly helpful results.

$$q + \tau 2q_t + \frac{\tau^2}{2}q_{tt} + \ldots = (\alpha - \beta)q - (\beta + \alpha)\sigma q_x + \frac{(\alpha - \beta)\sigma^2}{2}q_{xx} - \ldots \tag{17}$$

Now remember that α is not equal to 1, because this would not be a random walk if it were. Then to balance the q terms, $(1 - \alpha + \beta)q = 0$ implies that $q = 0$ at leading order. To continue on this, take the x derivative of 17 and see what survives.

It is interesting to note that when $\alpha = 1 - \epsilon$ ($\alpha \to 1$), this gives the Telegrapher's equation (18), which is diffusive at long times.

$$\frac{\partial^2 p}{\partial t^2} + \frac{q}{\tau_c}\frac{\partial p}{\partial t} = \nu^2 \frac{\partial^2 p}{\partial x^2} \tag{18}$$