Lecture 1: Theory of Financial Risks: A Physicist's Perspective (Overview)

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1 Motivations

1.1 Risk Control

Recently, Risk control has become a major concern of the finance industry. Thus simple Gaussian model is not satisfactory anymore and more reliable models for price fluctuations are required. Actually, many adequate tools to deal with these non Gaussian models are being developed now. (derivative pricing, portfolio management, etc.)[1]

1.2 Complex problems

However, the complexity of the financial market is so significant that it is even compared with the velocity field of a turbulent flow. Thus, simple models of price fluctuations fail and complex instruments (Derivatives, interest rate curve, etc.) are implemented. Eventually, finance is becoming an experimental science, like astronomy. You observe the data, use models to predict

[1] If you are interested in this introductory lecture, try to look up [1].

and try to explain them. Thus, it even requires a basic ingredient of human psychology.

2 Statistics of price fluctuations

2.1 Price model

In classical finance, price increments are independent on arbitrary small time scales. But even on most liquid markets, price increments are correlated over at least 10 minutes. (However, after 10 minutes, the correlation becomes near 0, turning into pure noise.) If the variance is finite, the only possibility is therefore the Gaussian (Central Limit Theorem): this is Bachelier's model (1900). However, we know that in Gaussian model, tails are ridiculously underestimated.

If the variance is infinite, one can have Lévy stable (power-law) distributions (Mandelbrot '63). But tails are now overestimated. Moreover, it is believed in the financial market that the variance is finite. The distribution of price increments on short time scales δx_i are power-law distributed:

$$P(\delta x_i) \sim \frac{A_i}{\| \delta x_i \|^{1+\mu}}, \quad \mu = 3 - 5 > 2 \tag{1}$$

Note that the 'Value-at-Risks' is the loss level corresponding to a certain (small) probability. In estimating VaR, the main concern is at tails. In a Gaussian world, the VaR of a portfolio comes from the 'conspiration' of small unrelated events. In a power-law world, one large event dominates. To estimate VaRs of complicated portfolio in power-law models, methods such as 'saddle-point method' are used.

2.2 Pitfalls

The problem is, the simple convolution rule does not work well:

$$x(T) - x_0 = \sum_{i=1}^{N} \delta x_i + IID \rightarrow P_T(\delta x) = [P_\tau(\delta x)]^{*N} \tag{2}$$

But the Kurtosis decays slower than $1/N$ (as $1/N^\alpha, \quad \alpha \sim 0.3$). There are long range 'volatility' correlations which slows down the convergence to the Gaussian.

$$< \delta x_i^2 \delta x_j^2 >_c \sim \| i - j \|^{-\alpha} \tag{3}$$

This explains the slow decay of the Kurtosis and the slow convergence towards the Gaussian.

- Note 1: Remember that the Central Limit Thoerem never holds in the tails, corresponding to large risks.
- Note 2: Long range power-law correlations in the volatility can explain an *apparent* 'multifractal' behaviour, i.e.:

$$<\| \delta x(T) \|^q> \propto T^{\zeta_q}, \quad , \zeta_q \neq q\zeta_1 \tag{4}$$

- Note 3: Return-Volatility correlations $< \delta x_i^2 \delta x_j^2 >$ $(j > i)$ are skewed.

However, market data are available(High frequency time series on thousands of assets). Thus, finance is becoming an 'experimental' science(cf. Response function of markets measured by Prediction Company).

2.3 The Agenda (for a physicist)

- *Observe* and describe (model building) rather than postulate.
- *Use* the models to predict and compare with observations.
- *Explain:* find 'microscopic' market models to explain the observations (herding, panic, auto-adaptive strategies, etc...) or create *artificial markets* for simulation.

3 Consequences for option pricing

3.1 Option pricing

An option is an insurance against a fall or rise of the value of an asset(call or put options).

However, there are some questions that should be pursued. What is the price C of the contract(option valuation)?[2] How to hedge this contract (i.e. minimize the risk associated with writing this contract)?

Let's consider *the Black-Scholes answer* (1973) for this.

If $x(t)$ is a Gaussian random process in continuous time, then:

- One can find a *perfect* hedging strategy $\phi(x,t)$ such that the option is *zero risk(!!!)*
- The price C of the option is known in terms of the volatility σ but independent of the mean return.

But the zero risk property does not hold as soon as time is discretized and/or the process is not Gaussian.

- Zero *is not a good approximation for anything.*
- The price becomes fuzzy: depends on risk aversion.
- There are systematic differences between the theoretical prices and the observed (market) prices.

3.2 More general theories

For more general stochastic processes, one can find an *optimal strategy* $\phi(x,t)$ which minimizes the risk.

- The risk only vanishes in the continuous time, Gaussian limit, and is in general a substantial fraction of the option price itself.
- In general, the price C depends on the mean return.
- For 'weekly' non-Gaussian processes, one can obtain a *cumulant expansion* as:

$$C(x_0, x_s, T) = C_G(x_0, x_s, T) + \kappa_T f(x_0, x_s, T) + ... \tag{5}$$

where κ_T is the kurtosis of the process on time scale T.

[2] Note that in reality, organized options market (with standardized strikes and maturities) allow one to trade options much like stock. That means, there are *market prices* for options.

- **The volatility smile:** the market practice is to use C_G with a non constant volatility σ, called the 'implied volatility'. Here, we get:

$$\sigma_{imp.}(x_s, T) = \sigma\Big[1 + \frac{\kappa_T}{24}\Big(\frac{(x_s - x_0)^2}{\sigma^2 T} - 1\Big)\Big] \tag{6}$$

  This can be used to obtain an 'implied' Kurtosis and compare it to the historical Kurtosis. Traders on option markets have learned to correct the Black-Scholes theory to account for fat-tails and volatility fluctuations.

4 Remarks on portfolio optimization

Reducing the global risk of a portfolio requires the knowledge of *correlations* between the assets returns:

$$C_{ab} =< \delta x_a(t) \delta x_b(t) > \tag{7}$$

However, C_{ab} contains a lot of noise: 95% of the eigenvalues spectrum is explained by *Random matrix Theory [2]*, in particular low-lying (low risk) ones.

Some portfolio optimization problems are equivalent to a spin-glass problem[3]. For example, find the weights p_a such that the risk $\sum_{a,b} p_a C_{ab} p_b$ is minimized, with a 'margin call' constraint $\sum_a \parallel p_a \parallel = 1$.

This is equivalent to finding the ground-state of a spin-glass with interaction matrix C_{ab}^{-1}. Typically, one has an exponentially large number of solutions.

References

[1] Jean-Philippe Bouchaud(2000), *Theory of Financial Risks*, 1st ed. Cambridge University press

[2] Laurent Laloux, Pierre Cizeau, Jean-Philippe Bouchaud, *Noise dressing of financial correlation matrices*, Physical Review Letters 83(7), 1467 (1999)

[3] Stefano Galluccio, Jean-Philippe Bouchaud, Marc Potters (CEA-Saclay and Science et Finance), *Rational decisions, random matrices and spin glasses*, Physica A 259,449 (1998)