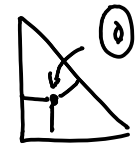
Recall: (X,J,ω) Kähler $\supset D$ anticanonical divisor, $\Omega \in \Omega^{n,0}(X-D)$ holom. vol. form

$$M = \left\{ (L,\nabla) \;/\; \begin{array}{l} L\subset X-D \text{ SLag torus} \\ \nabla \text{ flat } U(1) \text{ conn. on } \underline{\mathbb{C}}\to L \end{array} \right\}$$

$$W(L,\nabla) = \sum_{\substack{\beta\in\pi_2(X,L)\\ \mu(\beta)=2}} n_\beta(L)\, z_\beta(L,\nabla) \quad \text{where} \begin{cases} n_\beta(L) = \#\text{ holom. discs through } p\in L \text{ generic point} \\ z_\beta(L,\nabla) = \exp(-\int_\beta \omega)\cdot \mathrm{hol}_\nabla(\partial\beta) \end{cases}$$

Ex: $X = \mathbb{CP}^2$, $D = \{x_0x_1x_2 = 0\}$, $M = $ (subset of) $(\mathbb{C}^*)^2$

$W = z_1 + z_2 + e^{-\int_{\mathbb{CP}^1}\omega}/z_1z_2$



Today: 1. Motivation / interpretation
2. Wall-crossing & instanton corrections
3. Mirror symmetry for X vs. mirror symm. for D.

Interpretation: take $L = S^1(r_1)\times S^1(r_2) \subset (\mathbb{C}^*)^2 \subset \mathbb{CP}^2$.

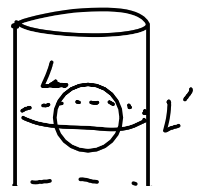
in $(\mathbb{C}^*)^2$, L doesn't bound any holom. discs, so $HF^*(L,L) \simeq H^*(T^2)$

$(L,\nabla) \longleftrightarrow$ a point in mirror $(\mathbb{C}^*)^2$

$Ext^*(\mathcal{O}_p,\mathcal{O}_p) \cong H^*(T^2)$

But in $\mathbb{CP}^2$, 1) L bounds holom. discs $\Rightarrow$ Floer theory is obstructed!

Ex:



: Floer diff. on $CF^*(L,L')$ has $\partial^2 \neq 0$!

Obstruction encoded by $m_0 \in CF^*(L,L)$ – count of discs (FO^3)

on $CF^*(L,L')$: $\partial^2(x) = m_0(L')\cdot x - x\cdot m_0(L)$

We're in "weakly unobstructed" case where m_0 is a multiple of the unit – in fact: $m_0 = W\cdot 1$.

Then can define $HF^*((L,\nabla),(L',\nabla'))$ only if $W(L,\nabla) = W(L',\nabla')$.

2) moreover: $HF(L,L)$ well defined, but typically <u>zero</u> ($\Rightarrow L \simeq 0$ in $DFuk$)
Indeed, except for Clifford torus, all product tori in $\mathbb{CP}^2$ are displaceable.

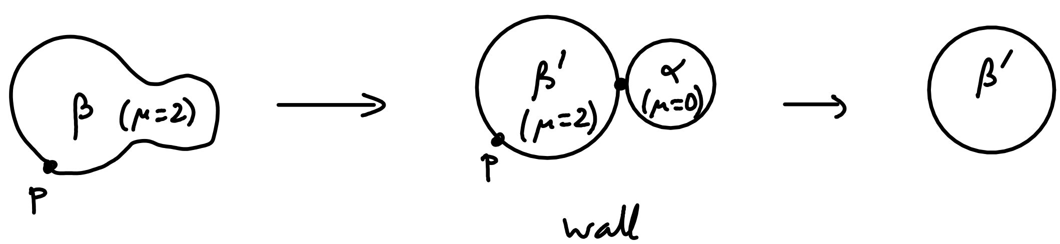
- To replicate this on the mirror, introduce superpotential W & replace $D^bCoh(M)$ by $D^b_{sing}(M,W) = \coprod_{\lambda\in\mathbb{C}} D^bCoh(W^{-1}(\lambda))/Perf\, W^{-1}(\lambda)$ (Orlov)

  ↑ "measures" singularities of $W^{-1}(\lambda)$

In particular:
- different level sets of W don't "see" each other
  [in terms of matrix fact^ns: $\mathcal{E} \in MF(W-\lambda)$, $\mathcal{E}' \in MF(W-\lambda')$
  $\Rightarrow$ diff^l on $hom(\mathcal{E},\mathcal{E}')$ squares to $(\lambda-\lambda')\,Id$]
- $p\in M \rightsquigarrow \mathcal{O}_p \simeq 0$ in D^b_{sing} unless $p \in Crit\, W$.

<u>Wall-crossing</u>: recall exp dim $\mathcal{M}(L,\beta) = n-3+\mu(\beta)$, $\mu(\beta) = 2(\beta\cap D)$

- in dim. $n=2$, expected dim. <0 for $\mu(\beta)=0$ ... generically L shouldn't bound any $\mu=0$ discs. However, in codim. 1 L bounds such a disc; this defines a "wall" in M, across which n_β (& hence W) jumps.



typically: $n_\beta = 1$, $n_{\beta'} = 1$ ... $n_\beta = 0$, $n_{\beta'} = 1$

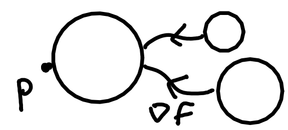
- in dim. $n\geq 3$, a generic L may bound $\mu=0$ discs, so $ev_*[\overline{\mathcal{M}}_1(L,\beta)]^{vir}$ is an <u>n-chain</u> on L, <u>not</u> a cycle (& depends on auxiliary data).

So $n_\beta(L)$ depends on choice of point $p\in L$... W multivalued!

Can lift ambiguity by choosing extra data, e.g.:

or
- the point p
- a Morse f^n $f: L \to \mathbb{R}$, to "complete" $ev_* \mathcal{M}_1(L,\beta)$ to a <u>cycle</u> by considering not only discs but also "clusters" [Cornea-Lalonde]

or
- ...



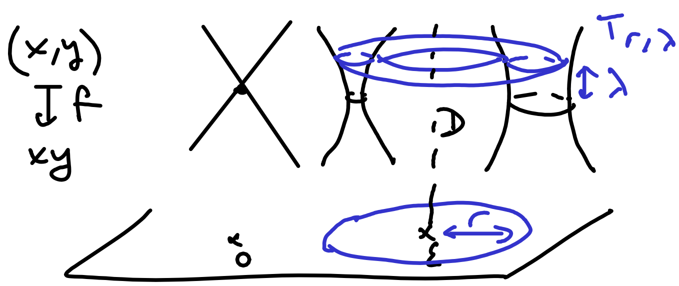
In any case, W is now a function of $(L, \nabla$, extra data$)$ and we still have discontinuities when n_β jumps across walls.

<u>Example</u>: $X = \mathbb{CP}^2$, $D = \{xy = \varepsilon\} \cup \{\text{line at } \infty\}$, $\Omega = \dfrac{dx \wedge dy}{xy - \varepsilon}$, ω standard

S^1 acts by $e^{i\theta} \cdot (x,y) \mapsto (xe^{i\theta}, ye^{-i\theta})$, moment map $\mu \left(= \frac{1}{2} \frac{|x|^2 - |y|^2}{1 + |x|^2 + |y|^2} \right)$

S^1-invt family of special Lagrangian tori:

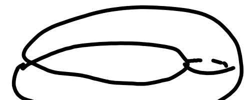
$$T_{r,\lambda} = \{ |xy - \varepsilon| = r, \ \mu = \lambda \}$$



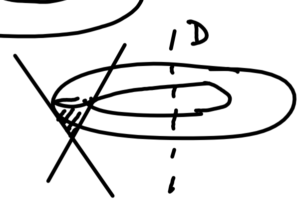
<u>Check</u>: $T_{(x,y)} T_{r,\lambda} = \text{span}\big((ix, -iy), v_2 \big)$, $v_1 = (ix,-iy)$ ↑ action of S^1 $(\partial/\partial\theta)_*$

- $\omega(v_1, v_2) \underset{def}{=} d\mu(v_2) = 0$ $(\mu = \text{cst})$ $\Rightarrow$ Lagr.
- $\iota_{v_1} \Omega = i\, d \log (xy - \varepsilon)$ which is real on v_2 $(|xy - \varepsilon| = \text{cst})$

- $T_{|\varepsilon|, 0}$ singular



- $T_{|\varepsilon|, \lambda}$ bound $\mu = 0$ discs



$\Rightarrow$ wall-crossing.

<u>For $r > |\varepsilon|$</u>: $T_{r,\lambda}$ deforms to product T^2 (shift circle $\varepsilon \rightsquigarrow 0$) without wall-crossing $\rightsquigarrow$ bounds 3 families of discs as before.

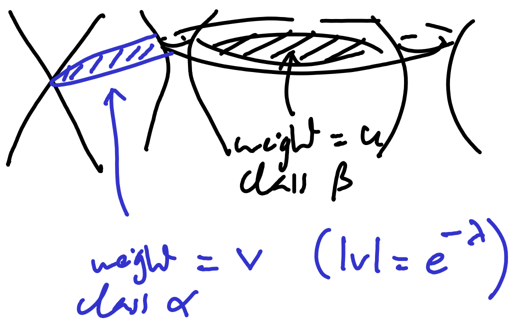
two are sections of f over $D(\varepsilon, r)$, classes β_1, β_2; 3^{rd} goes through line at ∞ class $[\mathbb{CP}^1] - \beta_1 - \beta_2$

$$W = z_1 + z_2 + \frac{e^{-A}}{z_1 z_2} \quad (A = \text{area}(\mathbb{P}^1))$$

<u>For $r<|\varepsilon|$</u>: $T_{r,\lambda}$ deforms to Chekanov-type torus $T_{r,0}$
bounds 4 families of discs (Chekanov-Schlenk, Polterovich ...)

one is a section of f over $D(\varepsilon,r)$, 3 others through line at ∞

$$n_\beta = 1,\quad n_{[\mathbb{CP}^1]-2\beta\pm\alpha} = 1,\quad n_{[\mathbb{CP}^1]-2\beta} = 2$$



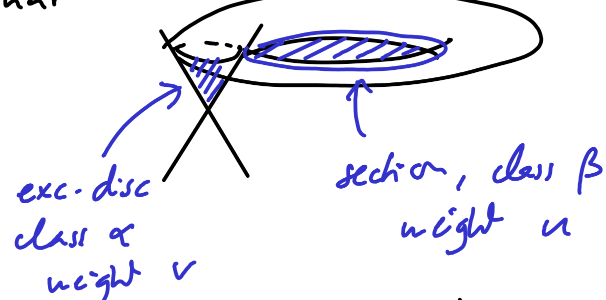
$$\Rightarrow W = u + \frac{e^{-A}(1+v)^2}{u^2 v}$$

As we cross $r=|\varepsilon|$ for $\lambda>0$, the disc in class β becomes class β_2,

so geometry of M would give $u \longleftrightarrow z_2$, $v \longleftrightarrow \frac{z_1}{z_2}$,

but then W is discontinuous across wall.

Instead, notice that



smooths to disc in class $\beta+\alpha=\beta_1$ on side $r>|\varepsilon|$ (<u>not</u> on side $r<|\varepsilon|$)

(& vice versa for discs through line at infinity)

So: want to correct: $\begin{cases} u \longleftrightarrow z_2(1+v) = z_2+z_1 \\ v \longleftrightarrow z_1/z_2 \end{cases}$

Then W is <u>continuous</u> & analytic across wall!

(similarly for portion $\lambda<0$ of wall: exceptional disc in class $-\alpha$ corrects gluing $u \longleftrightarrow z_1$ to $u \longleftrightarrow z_1(1+v^{-1}) = z_1 + z_2 v$)

<u>General wall-crossing result:</u> (FOOO):

> across wall, $\exists$ analytic change of variables on M that makes W continuous.

More precisely, when wall corresponds to a Maslov 0 disc in class α, gluing is

$$z_\beta \mapsto z_\beta \cdot h(z_\alpha)^{[\partial\beta]\cdot[\partial\alpha]}$$

← in dim. 2 (in higher dim., $\langle \nu(\text{wall}), [\partial\beta] \rangle$)

where $h(z_\alpha) = 1 + O(z_\alpha) \in \mathbb{Q}[[z_\alpha]]$

(coefficient of z_α^d = contrib^n of d-fold covers of exceptional disc; in our example $h(v) = 1+v$).

$\rightsquigarrow$ corrected mirror = glue chambers of M by these changes of coordinates.

Mirror symmetry for the pair (X, D):

Folklore statement: "the fiber of $W: M \to \mathbb{C}$ is mirror to D"

★ $D \subset X$ carries a natural induced holom. volume form, $\Omega_D = \mathrm{Res}_D(\Omega)$ residue of Ω along D, satisfies $\Omega = \Omega_D \wedge d\log \sigma_D$ + bounded

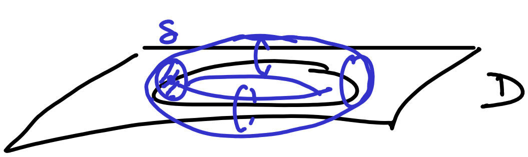
(σ_D ↑ defining sec^n of D)

★ Natural conjecture (supported by examples):

> the fibers of the SLag fibration $X \setminus D \xrightarrow{\pi} B$ near ∂B lie in a neighborhood of D, and are S^1-bundles over SLag tori in (D, ω_D, Ω_D)

(Ex: toric case; $T_{r,\lambda}$ for $r \to 0$ in above example)

In that case, ∂B = base of SYZ fibration on D....



- Fibers L of π near D bound a family of $\mu = 2$ meridian discs: small normal discs to D. Call δ the class of these discs.

Assume D smooth for simplicity & fibration π has the expected boundary structure. Then:

★ Near the boundary, $W = z_\delta + o(1)$ (all other discs have area $\gg$ area(δ))

★ $\partial M = \{|z_\delta| = 1\}$ (ie. area of meridian disc $\to 0$) L totally collapsed onto $\Lambda \subset D$

Actually ∂M is a bundle over S^1 with fiber $\underline{M_D := \{z_\delta = 1\}}$

The fibration is given by $\arg(z_\delta)$ = holonomy of ∇ along meridian

ie. $M_D = \left\{ \begin{array}{l} L \text{ collapsed onto } \Lambda \subset D \\ \nabla \text{ pulled back from flat conn on } \Lambda \end{array} \right\}$

Thus: $\underline{M_D = \text{SYZ mirror to } D}$! (uncorrected so far).

★ If we can neglect other terms in W (ie., in large vol. limit), then $M_D \cong \{W = 1\}$

Equivalently (recall $\omega \leftarrow \omega + t\, c_1(X)$ enlarges mirror):

> $\{W = e^t\}$ family of fibers, as $t \to \infty$, approximate family of SYZ mirrors to $(D, \omega_{|D} + t\, c_1(X)_{|D})$

However this only holds asymptotically, up to higher order terms.

2 reasons: 1) $W \neq z_\delta$ so $W^{-1}(1) \neq z_\delta^{-1}(1)$.

2) instanton corrections are different!

The geometry of $M_D \subset M$ gets corrected by wall-crossing in X — even in the collapsing limit $\to D$, Lagr. tori in X still cross walls when they bound $\mu = 0$ discs in X, vs. when the corresp. tori in D bound $\mu = 0$ discs in D.

★ On the other hand: all smooth fibers of W are symplectomorphic to each other and also to $\{z_\delta = 1\}$!

+ Kähler class unaffected by instanton corrections

$\Rightarrow$ expect:

> symplectically the fiber of W ($\simeq M_D$) is mirror to the divisor D.

HMS conjecture for pairs

[necessarily B-model on (X, D)
$\updownarrow$
A-model on (M, Λ_D)
the other side doesn't quite match...]

- **Relative Fukaya category** $\mathcal{F}(M, \Lambda_D)$: (recall $\Lambda_D = \{z_S = 1\} \subset \partial M$)

  objects = admissible Lagrangians $L \subset M$, $\partial L \subset \Lambda_D$ (poss. empty), + flat ∇. $z_{S|L} \in \mathbb{R}_+$ near ∂L.

  morphisms $\mathrm{Hom}(L_1, L_2) = CF(L_1, L_2^+)$ [Kontsevich, Seidel]
  ↳ perturb to positive direction

  ($\Leftrightarrow$ Fukaya cat. of LG-model $M \xrightarrow{z_S} D^2 \subset \mathbb{C}$, recall $W \sim z_S$ near ∂M, so expect $\mathcal{F}(M, \Lambda_D) \simeq \mathcal{F}(M, W)$).

- **Restriction (A_∞)-functor** $\rho: \mathcal{F}(M, \Lambda_D) \to \mathcal{F}(\Lambda_D)$

  on objects: $L \mapsto \partial L$

Conj. (HMS):

$$\begin{array}{ccc} D^b Coh(X) & \xrightarrow{\text{restr.}} & D^b Coh(D) \\ \text{HMS} \downarrow \simeq & & \simeq \downarrow \text{HMS} \\ D^\pi \mathcal{F}(M, \Lambda_D) & \xrightarrow{\text{restr.}} & D^\pi \mathcal{F}(D) \end{array}$$

Ex.: $\mathbb{CP}^2$ & Del Pezzo surfaces [A.-Katzarkov-Orlov]