Math 53 – Practice Final – Solutions

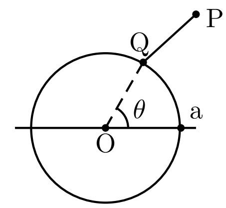
1. $P : (1, 1, -1)$, $Q : (1, 2, 0)$, $R : (-2, 2, 2)$, so $\overrightarrow{PQ} = \langle 0, 1, 1\rangle$ and $\overrightarrow{PR} = \langle -3, 1, 3\rangle$. Thus

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ -3 & 1 & 3 \end{vmatrix} = \langle 2, -3, 3\rangle.$$

The vector $\langle 2, -3, 3\rangle$ is normal to the plane through P, Q, R. Plugging any of the given points into the equation $2x - 3y + 3z = d$, we obtain:

$$2x - 3y + 3z = -4.$$

2. $\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$, where $\overrightarrow{OQ} = \langle a\cos\theta, a\sin\theta\rangle$, and $\overrightarrow{QP} = a\theta\,\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$.
Hence $x = a\cos\theta + \dfrac{a\theta}{\sqrt{2}}$, $y = a\sin\theta + \dfrac{a\theta}{\sqrt{2}}$.



3. a) $\vec{r} = \langle 3\cos t, 5\sin t, 4\cos t\rangle \Rightarrow \vec{v} = d\vec{r}/dt = \langle -3\sin t, 5\cos t, -4\sin t\rangle$, and $|\vec{v}| = \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t} = 5$.

b) The trajectory passes through the yz-plane when $x = 0$, hence when $\cos t = 0$, i.e. $t = \pi/2$ and $3\pi/2$. Thus, the intersections occur at the points $(0, \pm 5, 0)$.

4. $w = x^2y - xy^3$, and $P = (2, 1)$:

a) $\nabla w = \langle 2xy - y^3, x^2 - 3xy^2\rangle$, so $\nabla w(P) = \langle 3, -2\rangle$. The unit vector in the direction of $\vec{A} = \langle 3, 4\rangle$ is $\hat{u} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{\langle 3, 4\rangle}{5}$. So $D_{\hat{u}}w = \nabla w \cdot \hat{u} = \langle 3, -2\rangle \cdot \dfrac{\langle 3, 4\rangle}{5} = \dfrac{1}{5}$.

b) $D_{\hat{u}}w = \dfrac{1}{5} \approx \dfrac{\Delta w}{\Delta s}$, so $\Delta w \approx \dfrac{1}{5}\Delta s = \dfrac{1}{5}(0.01) = 0.002$.

5. a) Let $g(x, y, z) = x^2 + 2y^2 + 2z^2$: then $\nabla g = \langle 2x, 4y, 4z\rangle = \langle 2, 4, 4\rangle$ at $(1, 1, 1)$. Since ∇g is normal to the tangent plane, we get the equation: $2x + 4y + 4z = 10$, or $x + 2y + 2z = 5$.

b) Dihedral angle (angle between normals): $\cos\theta = \dfrac{\langle 1, 2, 2\rangle \cdot \langle 0, 0, 1\rangle}{(\sqrt{1^2 + 2^2 + 2^2})(1)} = \dfrac{2}{3}$. So $\theta = \cos^{-1}(2/3)$.

6. $f(x, y) = x^2 + xy + y^2 - 4x - 5y + 7$: the critical points correspond to $f_x = 2x + y - 4 = 0$ and $f_y = x + 2y - 5 = 0$. Solving, we get $x = 1$ and $y = 2$. So the only critical point is $(1, 2)$. Moreover $f(1, 2) = 0$.

Next we check the boundaries and infinity. On the x-axis: $f(x, 0) = x^2 - 4x + 7 = (x - 2)^2 + 3 > 0$; on the y-axis: $f(0, y) = y^2 - 5y + 7 = (y - \frac{5}{2})^2 + \frac{3}{4} > 0$. At infinity: if x and/or y tends to $+\infty$ then $f(x, y) \to +\infty$. So the minimum of f in the first quadrant is at $(1, 2)$.

(Note: the second derivative test shows that $(1, 2)$ is a local minimum, but this is not sufficient to conclude regarding the absolute minimum.)

7. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ with constraint $g(x, y, z) = 2x + y - z = 6$: the Lagrange equations ($\nabla f = \lambda\nabla g$) are: $2x = 2\lambda$, $2y = \lambda$, $2z = -\lambda$. Substituting into the constraint equation: $2x + y - z = 2\lambda + \frac{\lambda}{2} - (-\frac{\lambda}{2}) = 3\lambda = 6$. So $\lambda = 2$, and $(x, y, z) = (2, 1, -1)$.

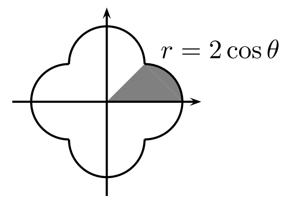
8. At the point P, differentiating the constraint gives: $dg = g_x\,dx + g_y\,dy + g_z\,dz = 0$, so $dz = -\dfrac{g_x}{g_z}\,dx - \dfrac{g_y}{g_z}\,dy$. Hence $\dfrac{\partial z}{\partial x} = -\dfrac{g_x}{g_z} = -\dfrac{2}{-1} = 2$ (using the given values of g_x and g_z.)

9. The region of integration is bounded by the parabola $y = x^2$ (or $x = \sqrt{y}$), the horizontal line $y = 9$, and the y-axis. So:

$$\int_0^3 \int_{x^2}^9 xe^{-y^2}\, dy\, dx = \int_0^9 \int_0^{\sqrt{y}} xe^{-y^2}\, dx\, dy.$$

Inner: $\left[\frac{1}{2}x^2 e^{-y^2}\right]_0^{\sqrt{y}} = \frac{1}{2}ye^{-y^2}$. Outer: $\left[-\frac{1}{4}e^{-y^2}\right]_0^9 = \frac{1}{4}(1 - e^{-81})$.

10. By symmetry, we integrate over 1/8 of the region; recalling that the polar moment of inertia is $I_0 = \iint_R r^2 \rho\, dA$ (here $\rho = 1$), we get



$$8 \int_0^{\pi/4} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta \qquad \left(\text{or } 4 \int_{-\pi/4}^{\pi/4} \cdots\right)$$

11. a) $\hat{\mathbf{n}}\, ds = \langle dy, -dx\rangle$, so Flux $= \int_C \vec{F} \cdot \hat{\mathbf{n}}\, ds = \int_C -Q\, dx + P\, dy$.

b) By Green's theorem, $\oint_C -Q\, dx + P\, dy = \iint_R (P_x + Q_y)\, dA = \iint_R (a+b)\, dA = (a+b)\,\text{area}(R)$. This equals the area of R if and only if $a + b = 1$.

12. $\bar{z} = \dfrac{1}{\text{Volume}} \iiint z\, dV$, since here the density is 1. The volume is $\frac{2}{3}\pi$ (half of the unit sphere), and in spherical coordinates $z = \rho\cos\phi$. So $\bar{z} = \dfrac{3}{2\pi} \displaystyle\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

Inner: $\int_0^1 \rho^3 \cos\phi \sin\phi\, d\rho = \left[\frac{1}{4}\rho^4 \cos\phi\sin\phi\right]_0^1 = \frac{1}{4}\cos\phi\sin\phi$.
Middle: $\int_0^{\pi/2} \frac{1}{4}\sin\phi\, \cos\phi\, d\phi = \left[\frac{1}{8}\sin^2\phi\right]_0^{\pi/2} = \frac{1}{8}$. Outer: $\frac{3}{2\pi} \cdot 2\pi \cdot \frac{1}{8} = \frac{3}{8}$.

13. The line from $P : (1,1,1)$ to $Q : (2,4,8)$ has parametric equations: $x = 1 + t$, $y = 1 + 3t$, $z = 1 + 7t$ (since $\overrightarrow{PQ} = \langle 1, 3, 7\rangle$). The line segment corresponds to $0 \le t \le 1$. So

$\int_C (y - x)\, dx + (y - z)\, dz = \int_0^1 2t\, dt + (-4t)\, 7\, dt = \int_0^1 -26t\, dt = \left[-13t^2\right]_0^1 = -13$.

14. a) $\vec{F} = \langle P, Q, R\rangle = \langle ay^2, 2yx + 2yz, by^2 + z^2\rangle$: we need $P_y = 2ay = Q_x = 2y$, so $a = 1$; and $P_z = 0 = R_x$; and $Q_z = 2y = R_y = 2by$, so $b = 1$. So: $\vec{F}$ is conservative when $a = 1$ and $b = 1$.

b) $f_x = y^2$, so $f(x,y,z) = xy^2 + g(y,z)$. Differentiating wrt y, $f_y = 2xy + g_y = 2xy + 2yz$.
So $g_y = 2yz$, hence $g(y,z) = y^2 z + h(z)$ and $f(x,y,z) = xy^2 + y^2 z + h(z)$.
Differentiating wrt z, $f_z = y^2 + h'(z) = y^2 + z^2$ so $h'(z) = z^2$, hence $h(z) = \frac{1}{3}z^3 + c$.
Finally we get: $f(x,y,z) = xy^2 + y^2 z + \frac{1}{3}z^3 + c$.

c) any surface S of the form $xy^2 + y^2 z + \frac{1}{3}z^3 = K$ for some constant K (i.e. a level surface of f). Then by the fundamental theorem, $\int_P^Q \vec{F} \cdot d\vec{r} = f(Q) - f(P) = 0$ if P and Q lie on S.

15. $\displaystyle\iint_B \vec{F} \cdot \hat{\mathbf{n}}\, dS + \iint_U \vec{F} \cdot \hat{\mathbf{n}}\, dS = \iint_S \vec{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_R \text{div}\, \vec{F}\, dV = \iiint_R 3\, dV = 3\, vol(V)$, where

$$vol(V) = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r\, dz\, dr\, d\theta = 2\pi \int_0^1 (1 - r^2)\, r\, dr = 2\pi \left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 = \pi/2.$$

So the total flux through B and U equals $3\pi/2$. Next we compute directly the flux through the bottom disc B, where $z = 0$ and $\hat{\mathbf{n}} = -\hat{\mathrm{k}}$:
$\iint_B \vec{F} \cdot \hat{\mathbf{n}}\, dS = \iint_B \langle x, y, z\rangle \cdot \langle 0, 0, -1\rangle\, dx\, dy = \iint_B -z\, dx\, dy = \iint_B 0\, dS = 0$.

Hence $\iint_U \vec{F} \cdot \hat{\mathbf{n}}\, dS = \iint_S \vec{F} \cdot \hat{\mathbf{n}}\, dS = 3\pi/2$.

16. U is the graph $z = f(x,y) = 1 - x^2 - y^2$, so $\hat{\mathbf{n}}\,dS = \langle -f_x, -f_y, 1\rangle dx\,dy = \langle 2x, 2y, 1\rangle dx\,dy$.

So $\displaystyle\iint_U \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iint_U \langle x,y,z\rangle\cdot\langle 2x,2y,1\rangle dx\,dy = \iint_U (2x^2+2y^2+z)\,dx\,dy.$

Recalling that $z = 1 - x^2 - y^2$ on U, this is equal to

$$\iint_U (x^2+y^2+1)\,dx\,dy = \int_0^{2\pi}\int_0^1 (r^2+1)\,r\,dr\,d\theta = 2\pi\left[\tfrac{1}{4}r^4 + \tfrac{1}{2}r^2\right]_0^1 = 2\pi\cdot\frac{3}{4} = 3\pi/2.$$

17. By Stokes, if S_1 is the portion of S enclosed by C, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_{S_1}(\nabla\times\vec{F})\cdot\hat{\mathbf{n}}\,dS$. Here

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & xz \end{vmatrix} = \langle 0, -z, 0\rangle, \quad \text{and } \hat{\mathbf{n}}\,dS = \left\langle -\tfrac{\partial z}{\partial x}, -\tfrac{\partial z}{\partial y}, 1\right\rangle dx\,dy = \langle -f'(x), 0, 1\rangle dx\,dy$$

(since S_1 is a graph $z = f(x)$). So $\oint_C \vec{F}\cdot d\vec{r} = \iint_{S_1}(\nabla\times\vec{F})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} 0\,dx\,dy = 0.$

18. a) C is the circle $x^2+y^2=1$, $z=1$. Parametrization: $x=\cos t$, $y=\sin t$, $z=1$; and $dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = 0$. So

$I = \oint_C xz\,dx + y\,dy + y\,dz = \int_0^{2\pi}\cos t(-\sin t)\,dt + \sin t\cos t\,dt + 0\,dt = 0.$

b) $$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & y & y \end{vmatrix} = \hat{\imath} + x\hat{\jmath}.$$

c) By Stokes' theorem, $I = \oint_C \vec{F}\cdot d\vec{r} = \iint_S(\nabla\times\vec{F})\cdot\hat{\mathbf{n}}\,dS = \iint_S(\hat{\imath}+x\hat{\jmath})\cdot\hat{\mathbf{n}}\,dS.$

Note: $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\langle x,y,z\rangle$, so $\vec{F}\cdot\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(x+xy)$ (where $x = \sqrt{2}\sin\phi\cos\theta$ and $y = \sqrt{2}\sin\phi\sin\theta$); and $dS = 2\sin\phi\,d\phi\,d\theta$.