Math 53 – Practice Midterm 1 A – 80 minutes

Problem 1. (10 points)

Find the area enclosed by a loop of the curve given by the polar equation $r = \sqrt{\sin 2\theta}$.

Problem 2. (15 points)

a) Find the area of the space triangle with vertices $P_0 : (2, 1, 0)$, $P_1 : (1, 0, 1)$, $P_2 : (2, -1, 1)$.

b) Find the equation of the plane containing the three points P_0, P_1, P_2.

c) Find the intersection of this plane with the line parallel to the vector $\vec{V} = \langle 1, 1, 1 \rangle$ and passing through the point $S : (-1, 0, 0)$.

Problem 3. (15 points)

a) Let $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be the position vector of a path. Give a simple intrinsic formula for $\frac{d}{dt}(\vec{r} \cdot \vec{r})$ in vector notation (not using coordinates).

b) Show that if $\vec{r}$ has constant length, then $\vec{r}$ and $\vec{v}$ are perpendicular.

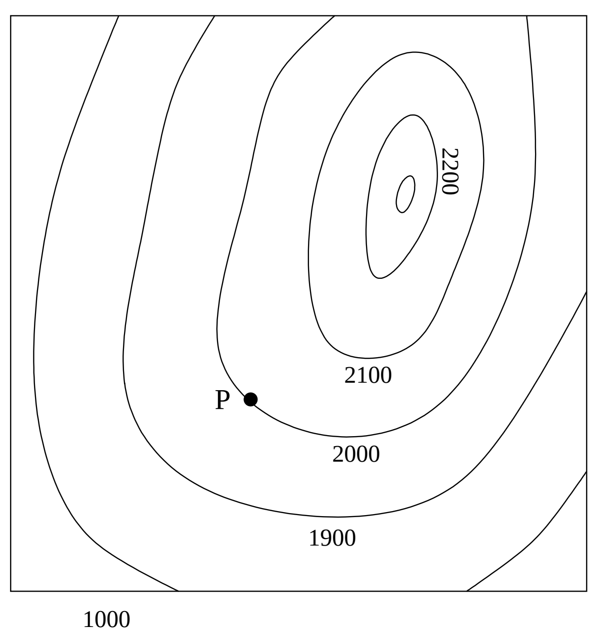
c) let $\vec{a}$ be the acceleration: still assuming that $\vec{r}$ has constant length, and using vector differentiation, express the quantity $\vec{r} \cdot \vec{a}$ in terms of the velocity vector only.

Problem 4. (10 points)

On the topographical map below, the level curves for the height function $h(x, y)$ are marked (in feet); adjacent level curves represent a difference of 100 feet in height. A scale is given.

a) Estimate to the nearest .1 the value at the point P of the directional derivative $D_{\hat{u}}h$, where $\hat{u}$ is the unit vector in the direction of $\hat{\imath} + \hat{\jmath}$.

b) Mark on the map a point Q at which $h = 2200$, $\frac{\partial h}{\partial x} = 0$ and $\frac{\partial h}{\partial y} < 0$. Estimate to the nearest .1 the value of $\frac{\partial h}{\partial y}$ at Q.



Problem 5. (10 points)

Let $f(x,y) = xy - x^4$.

a) Find the gradient of f at $P:(1,1)$.

b) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x,y)$ at the point $(x,y) = (1,1)$.

Problem 6. (5 points)

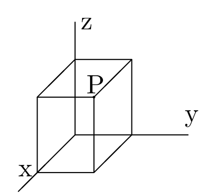
Find the equation of the tangent plane to the surface $x^3y + z^2 = 3$ at the point $(-1,1,2)$.

Problem 7. (5 points)

Let $w = f(u,v)$, where $u = xy$ and $v = x/y$. Express $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.

Problem 8. (20 points)

A rectangular box is placed in the first octant as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P:(x,y,z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. Which P gives the box of greatest volume?



a) Show that the problem leads one to maximize $f(x,y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f.

b) Find a critical point of f which lies in the first quadrant $(x > 0,\ y > 0)$, and determine its nature by using the second derivative test.

c) Find the maximum of f in the first quadrant (justify your answer).

Problem 9. (10 points)

In Problem 8 above, instead of substituting for z, one could also use Lagrange multipliers to maximize the volume $V = xyz$ with the same constraint $x^2 + y^2 + z = 1$.

a) Write down the Lagrange multiplier equations for this problem.

b) Solve the equations (still assuming $x > 0$, $y > 0$).