Math 53 Midterm 2 Monday, Nov 21, 2011 1:10 - 2:00

Directions: Do all the work on these pages; use reverse side if needed. Answers without accompanying reasoning may only receive partial credit. **No books, notes, calculators, or electronic devices**. Please stop when asked to and don't talk until your paper is handed in.

YOUR NAME: SOLUTIONS

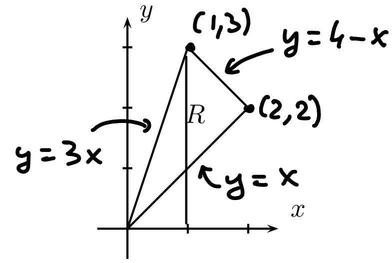
IMPORTANT: Please mark the box next to your GSI's name, and circle your discussion section's number/time.

Discussion section:				
	☐ Andreas VOELLMER	# 101 (8am),	# 106 (11am)	
	☐ Elan BECHOR	# 102 (8am),	# 105 (12:30),	# 120 (2pm)
	☐ Jae-young PARK	# 103 (8am),	# 114 (2pm)	
	☐ Alexandru CHIRVASITU	# 104 (12:30),	# 110 (5pm)	
	☐ Alex KRUCKMAN	# 107 (11am),	# 109 (12:30)	
	☐ Nam TRANG	# 108 (12:30),	# 113 (2pm)	
	☐ Christian HILAIRE	# 111 (5pm),	# 118 (8am)	
	☐ Natth BEJRABURNIN	# 112 (2pm),	# 117 (3:30)	
	☐ Kevin WRAY	# 115 (2pm),	# 119 (12:30)	
	☐ Yuhao HUANG	# 116 (6pm)		

GRADING	
1.	/25
2.	/20
3.	/15
4.	/25
5.	/15
TOTAL	/100

Problem 1. (25)

a) (10) Let R be the triangle with vertices $(0,0)$, $(1,3)$, and $(2,2)$ (see picture). Set up the integral $\iint_R \frac{(x+y)^3}{x^2}\,dA$ as an iterated integral. Give the bounds of integration, but <u>do not evaluate</u>.



$$\int_0^1\int_x^{3x} \frac{(x+y)^3}{x^2}\,dy\,dx + \int_1^2\int_x^{4-x}\frac{(x+y)^3}{x^2}\,dy\,dx$$

(split into two since the top boundary is given by $y=3x$ for $x<1$, $y=4-x$ for $x>1$.)

$$\left(\text{or: } \int_0^2\int_{y/3}^{y}\frac{(x+y)^3}{x^2}\,dx\,dy + \int_2^3\int_{y/3}^{4-y}\frac{(x+y)^3}{x^2}\,dx\,dy\right)$$

b) (15) Set up the integral of part (a) as an iterated integral in terms of the variables $u = y/x$ and $v = x+y$. Give the integrand and the bounds, but <u>do not evaluate.</u>

[If necessary, note that $x = v/(1+u)$.]

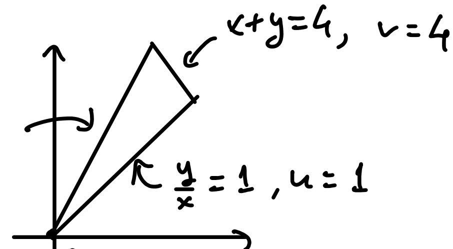
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -y/x^2 & 1/x \\ 1 & 1 \end{vmatrix} = -\frac{y}{x^2} - \frac{1}{x} = -\frac{(x+y)}{x^2}$$

so $du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy = \frac{(x+y)}{x^2}\,dx\,dy$, and so

$$\frac{(x+y)^3}{x^2}\,dx\,dy = (x+y)^2\,du\,dv = v^2\,du\,dv$$

Bounds: $\begin{cases} 1 \le u \le 3 \\ 0 \le v \le 4 \end{cases}$ since

$\frac{y}{x} = 3$
$u=3$

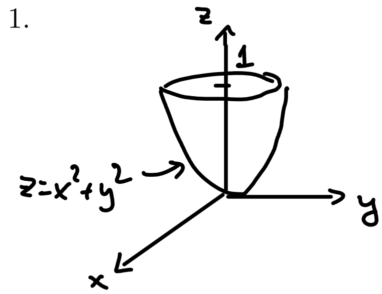


(at (0,0): u undefined, $v=0$)

Answer: $\int_1^3\int_0^4 v^2\,dv\,du$ (or $\int_0^4\int_1^3 v^2\,du\,dv$)

Problem 2. (20)

a) (10) Find the moment of inertia $I_z = \iiint (x^2 + y^2)\, dV$ of the solid (with uniform density 1) bounded below by the paraboloid $z = x^2 + y^2$, and above by the plane $z = 1$.



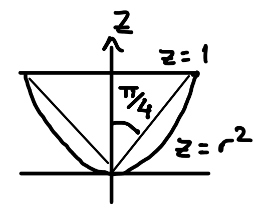
The paraboloid and the plane intersect when $x^2+y^2=1$ or $r=1$.

$$\text{So: } I_z = \iiint_R r^2\, dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 r^2\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^3 - r^5)\, dr\, d\theta$$

$$= 2\pi \left[\frac{r^4}{4} - \frac{r^6}{6}\right]_0^1 = 2\pi \cdot \frac{1}{12} = \frac{\pi}{6}.$$

b) (10) Set up (integrand and bounds, but <u>*do not evaluate*</u>) a triple integral in *spherical* coordinates giving the moment of inertia I_z of the solid in part (a).



The plane is $z = \rho\cos\phi = 1 \iff \rho = \frac{1}{\cos\phi}$

The paraboloid is: $z = r^2 \iff \rho\cos\phi = (\rho\sin\phi)^2 \iff \rho = \frac{\cos\phi}{\sin^2\phi}$

They intersect at $\phi = \frac{\pi}{4}$; hence upper bound for ρ is $1/\cos\phi$ for $\phi \le \frac{\pi}{4}$

$\cos\phi/\sin^2\phi$ for $\phi \ge \pi/4$.

Integrand: $(x^2+y^2)\,dV = r^2\, dV = \rho^2\sin^2\phi \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta$

$$\text{So: } I_z = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{1/\cos\phi} \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta$$

$$+ \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cos\phi/\sin^2\phi} \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta$$

Problem 3. (15) $\nearrow P$ $\nearrow Q$ $\nearrow R$

Let $\vec{F} = (az^2 + 2xy)\hat{\imath} + (x^2 + 2z)\hat{\jmath} + (2y + xz)\hat{k}$ (where a is a constant).

a) (6) Show that there is a certain value of a for which $\vec{F}$ is conservative.

$P_y = 2x = Q_x$;

$P_z = 2az$ and $R_x = z$ are equal for $\underline{a = 1/2}$

and $Q_z = 2 = R_y$.

b) (9) For this particular value of a, find a potential function for $\vec{F}$. Use a systematic method.

$$f_x = \frac{1}{2}z^2 + 2xy \underset{\int dx}{\Longrightarrow} f = \frac{1}{2}xz^2 + x^2y + g(y,z)$$

so $f_y = x^2 + g_y$; want $f_y = x^2 + 2z$, hence $g_y = 2z$

$$\underset{\int dy}{\Longrightarrow} g(y,z) = 2yz + h(z)$$

so $f(x,y,z) = \frac{1}{2}xz^2 + x^2y + 2yz + h(z)$

and $f_z = xz + 2y + h'(z)$; want $f_z = xz + 2y$, so $h'(z) = 0$

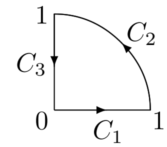
$\Rightarrow h = $ const,

$$f(x,y,z) = \frac{1}{2}xz^2 + x^2y + 2yz \quad (+\text{constant})$$

Problem 4. (25)

Let C_1 be a line segment from $(0,0)$ to $(1,0)$, C_2 an arc of the unit circle running from $(1,0)$ to $(0,1)$, and C_3 a line segment from $(0,1)$ to $(0,0)$ (see figure). Let C be the simple closed curve formed by C_1, C_2, C_3, and let $\vec{F} = -x^2\,\hat{\imath} + xy\,\hat{\jmath}$. Calculate the line integral $\oint_C \vec{F}\cdot d\vec{r}$:

a) (15) directly;



C_1: $x = t$, $y = 0$ $\quad dx = dt$, $dy = 0$ $\quad 0 \le t \le 1$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} -x^2 dx + xy\, dy = \int_0^1 -t^2 dt = -\frac{1}{3}.$$

C_2: $x = \cos t$, $y = \sin t$ $\quad dx = -\sin t\, dt$, $dy = \cos t\, dt$ $\quad 0 \le t \le \frac{\pi}{2}$

$$\int_{C_2} -x^2 dx + xy\, dy = \int_0^{\frac{\pi}{2}} -\cos^2 t(-\sin t)dt + \cos t \sin t \cos t\, dt$$

$$= \int_0^{\frac{\pi}{2}} 2\cos^2 t \sin t\, dt = \left[-\frac{2}{3}\cos^3 t\right]_0^{\pi/2} = \frac{2}{3}.$$

C_3: $x = 0$ so $\vec{F} = 0$, hence $\int_{C_3} \vec{F}\cdot d\vec{r} = 0$.

Total: $\oint_C \vec{F}\cdot d\vec{r} = -\frac{1}{3} + \frac{2}{3} + 0 = \frac{1}{3}$.

b) (10) using Green's theorem.

$$\oint_C -x^2 dx + xy\, dy = \iint_R \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(-x^2)\right) dA$$

$$= \iint_R y\, dA$$

$$= \int_0^{\pi/2}\int_0^1 r\sin\theta\, r\, dr\, d\theta$$

$$= \left(\int_0^{\pi/2} \sin\theta\, d\theta\right)\left(\int_0^1 r^2 dr\right)$$

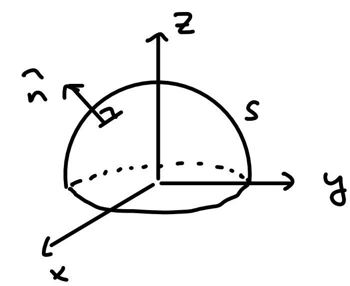
$$= (1)\left(\frac{1}{3}\right) = \frac{1}{3}.$$

(or: $\int_0^1 \int_0^{\sqrt{1-x^2}} y\, dy\, dx = \int_0^1 \frac{1-x^2}{2} dx = \left[\frac{x}{2} - \frac{x^3}{6}\right]_0^1 = \frac{1}{3}$).

Problem 5. (15)

Calculate the flux of $\vec{F} = z^4\,\hat{\mathrm{k}}$ out of the upper half ($z \geq 0$) of the unit sphere $x^2 + y^2 + z^2 = 1$.

sphere surface: (radius $a = 1$)



$$\hat{n} = \frac{\langle x, y, z\rangle}{a} = \langle x, y, z\rangle$$

$$dS = a^2 \sin\phi \, d\phi \, d\theta = \sin\phi \, d\phi \, d\theta$$

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \langle 0, 0, z^4\rangle \cdot \langle x, y, z\rangle \, dS$$

$$= \iint_S z^5 \, dS$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (\cos\phi)^5 \sin\phi \, d\phi \, d\theta \qquad (z = \cos\phi)$$

$$= 2\pi \cdot \left[-\frac{1}{6}\cos^6\phi\right]_0^{\pi/2} = +\frac{2\pi}{6} = \frac{\pi}{3}.$$

$\star$ Or, using divergence theorem:

total flux through $\begin{cases} S \\ \text{unit disk } T \text{ in } xy\text{-plane with } \hat{n} = -\hat{k} \text{ downwards} \end{cases}$

$$\iint_{S+T} \vec{F}\cdot\hat{n}\, dS = \iiint \operatorname{div}\vec{F}\, dV = \iiint 4z^3 \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 4(\rho\cos\phi)^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \cdot \left(\int_0^{\pi/2} 4\cos^3\phi \sin\phi \, d\phi\right)\left(\int_0^1 \rho^5 \, d\rho\right) = \frac{\pi}{3}$$

and $\iint_T \vec{F}\cdot\hat{n}\, dS = 0$ since $\vec{F} = 0$ on T $(z = 0)$

so $\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_{S+T} - \iint_T = \frac{\pi}{3} - 0 = \frac{\pi}{3}.$