Math 53 Midterm 1 Monday, Oct 10, 2011 1:10 - 2:00

Directions: Do all the work on these pages; use reverse side if needed. Answers without accompanying reasoning may only receive partial credit. **No books, notes, calculators, or electronic devices**. Please stop when asked to and don't talk until your paper is handed in.

YOUR NAME: SOLUTIONS

IMPORTANT: Please mark the box next to your GSI's name, and circle your discussion section's number/time.

Discussion section:	☐ Andreas VOELLMER	# 101 (8am),	# 106 (11am)	
	☐ Elan BECHOR	# 102 (8am),	# 105 (12:30),	# 120 (2pm)
	☐ Jae-young PARK	# 103 (8am),	# 114 (2pm)	
	☐ Alexandru CHIRVASITU	# 104 (12:30),	# 110 (5pm)	
	☐ Alex KRUCKMAN	# 107 (11am),	# 109 (12:30)	
	☐ Nam TRANG	# 108 (12:30),	# 113 (2pm)	
	☐ Christian HILAIRE	# 111 (5pm),	# 118 (8am)	
	☐ Natth BEJRABURNIN	# 112 (2pm),	# 117 (3:30)	
	☐ Kevin WRAY	# 115 (2pm),	# 119 (12:30)	
	☐ Yuhao HUANG	# 116 (6pm)		

GRADING	
1. ______	/20
2. ______	/20
3. ______	/20
4. ______	/20
5. ______	/20
TOTAL	/100

Problem 1. (20)

a) (7) Find an equation of the plane $\mathcal{P}$ through the points $A = (2,-1,0)$, $B = (3,2,1)$ and $C = (2,0,1)$.

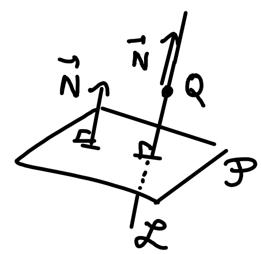
Normal vector: $\vec{N} = \vec{AB} \times \vec{AC} = \langle 1,3,1\rangle \times \langle 0,1,1\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \langle 2,-1,1\rangle$$

Equation: $\boxed{2x - y + z = 5}$

b) (8) Let $\mathcal{L}$ be the line through $Q = (3,-1,1)$ and perpendicular to $\mathcal{P}$. Find the point where $\mathcal{L}$ intersects $\mathcal{P}$.

The normal vector $\vec{N} = \langle 2,-1,1\rangle$ to $\mathcal{P}$ is parallel to $\mathcal{L}$, so $\mathcal{L}$ has parametric equations



$$\begin{cases} x = 3+2t \\ y = -1-t \\ z = 1+t \end{cases}$$

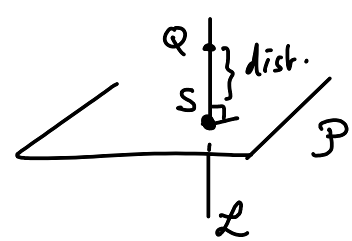
Plugging into equation of $\mathcal{P}$: $2x - y + z = 2(3+2t) - (-1-t) + (1+t)$

$$= 8 + 6t = 5$$

$\Rightarrow t = -\frac{1}{2}$; $\mathcal{L}$ intersects $\mathcal{P}$ at $\boxed{\left(2, -\frac{1}{2}, \frac{1}{2}\right)} = S$.

c) (5) Find the distance between the point Q and the plane $\mathcal{P}$.

The point of $\mathcal{P}$ closest to Q is the point S where $\mathcal{L}$ intersects $\mathcal{P}$ (see figure): So: $S = \left(2, -\frac{1}{2}, \frac{1}{2}\right)$.



Distance: $|\vec{QS}| = \sqrt{(2-3)^2 + \left(-\frac{1}{2}+1\right)^2 + \left(\frac{1}{2}\right)^2}$

$$= \sqrt{1 + \frac{1}{4} + \frac{1}{4}} = \boxed{\sqrt{\frac{3}{2}}}.$$

(Or, using distance formula: $\text{dist} = \dfrac{|ax+by+cz-d|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|2x-y+z-5|}{\sqrt{6}} = \dfrac{3}{\sqrt{6}} = \sqrt{\dfrac{3}{2}}$.)

Problem 2. (20) Consider the motion of a point in space given by the position vector

$$\vec{r}(t) = \langle t + \sin t, \sqrt{2}\cos t, t - \sin t\rangle, \quad 0 \le t \le 2\pi.$$

a) (10) Show that the speed is constant, and find the length of the trajectory.

$$\vec{v} = \frac{d\vec{r}}{dt} = \langle 1+\cos t, -\sqrt{2}\sin t, 1-\cos t\rangle$$

Speed: $|\vec{v}| = \sqrt{(1+\cos t)^2 + (-\sqrt{2}\sin t)^2 + (1-\cos t)^2}$

$$= \sqrt{(1+\cancel{2\cos t}+\cos^2 t) + 2\sin^2 t + (1-\cancel{2\cos t}+\cos^2 t)}$$

$$= \sqrt{2+2\cos^2 t + 2\sin^2 t}$$

$$= \sqrt{4} = \boxed{2} \quad \text{(independent of } t\text{)}$$

$$\text{Length} = \int_0^{2\pi} \frac{ds}{dt}\,dt = \int_0^{2\pi} |\vec{v}|\,dt = \int_0^{2\pi} 2\,dt = 2\cdot 2\pi = \boxed{4\pi}.$$

b) (10) Show that the velocity and acceleration vectors are perpendicular.

$$\vec{a} = \frac{d\vec{v}}{dt} = \langle -\sin t, -\sqrt{2}\cos t, +\sin t\rangle$$

$$\vec{a}\cdot\vec{v} = -\sin t(1+\cos t) + 2\sin t\cos t + \sin t(1-\cos t)$$

$$= 0.$$

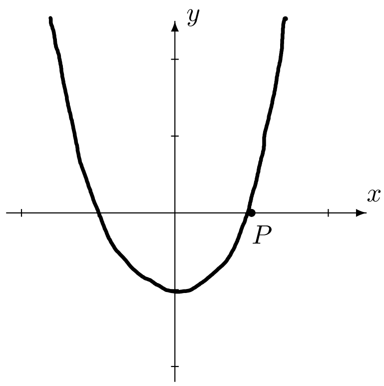
So $\vec{a} \perp \vec{v}$.

(<u>or:</u> $|\vec{v}| = \text{constant}$ by part (a) $\Rightarrow \frac{d}{dt}(\vec{v}\cdot\vec{v}) = 2\vec{v}\cdot\frac{d\vec{v}}{dt} = 0$

so $\vec{a} = \frac{d\vec{v}}{dt} \perp \vec{v}$)

Problem 3. (20) Let $f(x,y) = x^2 - y$.

a) (10) Sketch the level curve of f passing through the point $P : (1,0)$, and calculate the gradient vector ∇f at P.



$f(1,0) = 1$

$f(x,y) = 1 \iff x^2 - y = 1$

$\iff y = x^2 - 1.$

At P:

$\nabla f = \langle f_x, f_y \rangle = \langle 2x, -1 \rangle = \boxed{\langle 2, -1 \rangle}$

b) (10) Assume that a parametric curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ passes through the point P at $t = 0$, at unit speed, and the value of $f(x(t), y(t))$ reaches its maximum at $t = 0$. What can you say about the velocity vector at $t = 0$?

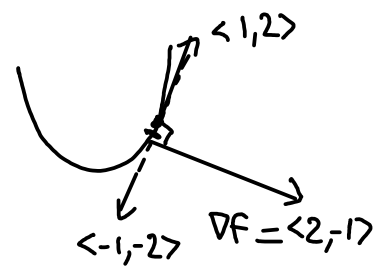
$$\frac{d}{dt}(f(x(t), y(t))) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = 0 \text{ at } t=0$$

At $t=0$, $\nabla f = \langle 2,-1 \rangle$, so the velocity vector $\vec{v} \perp \langle 2,-1 \rangle$.

(or: directional derivative $D_{\hat{v}} f = 0$ so $\hat{v} \perp \nabla f = \langle 2,-1 \rangle$).

So $\vec{v}$ is parallel to $\langle 1,2 \rangle$ (or to $\langle -1,-2 \rangle$)

∇f rotated $90°$



Also know $\vec{v}$ has unit length

$$\Rightarrow \boxed{\vec{v} = \frac{\langle 1,2 \rangle}{\sqrt{5}} \quad \text{or} \quad \vec{v} = \frac{\langle -1,-2 \rangle}{\sqrt{5}}}$$

Problem 4. (20)

a) (10) Find the critical points of $w = x^2 + 2xy + y^3 - 2y^2$, and determine their types.

$$\begin{cases} w_x = 2x + 2y = 0 & \Leftrightarrow \quad x = -y \\ w_y = 2x + 3y^2 - 4y = 0 & \Rightarrow \text{ plugging } x = -y, \end{cases}$$

$3y^2 - 6y = 0$

$y^2 - 2y = 0$

$y = 0$ or $y = 2$.

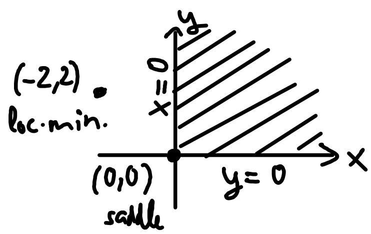
2 critical points: $\boxed{(0,0) \text{ and } (-2,2)}$.

$w_{xx} = 2, \quad w_{yy} = 6y - 4, \quad w_{xy} = 2$

At $(0,0)$, $D = 2 \cdot (-4) - (2)^2 = -12 < 0$: saddle.

At $(-2,2)$, $D = 2 \cdot 8 - (2)^2 = 12 > 0$ and $w_{xx} > 0$: local min.

b) (10) Find the point of the first quadrant ($x \geq 0$, $y \geq 0$) at which w is the smallest. Justify your answer.



- critical points:

$(0,0)$ is a saddle point (& not in interior of 1^{st} quadrant)

$(-2,2)$ is not in the first quadrant

- boundaries:

on x-axis $(y = 0, x \geq 0)$: $w(x,0) = x^2$ reaches its minimum at $x = 0$, $w(0,0) = 0$.

on y-axis $(x = 0, y \geq 0)$: $w(0,y) = y^3 - 2y^2$. reaches its minimum at $y = \frac{4}{3}$

(and $\to \infty$ as $y \to \infty$)

since $\frac{d}{dy}(y^3 - 2y^2) = 3y^2 - 4y$ and $3y^2 - 4y = 0 \Leftrightarrow y = 0$ or $y = \frac{4}{3}$.

$$w(0, \tfrac{4}{3}) = \left(\tfrac{4}{3}\right)^3 - 2\left(\tfrac{4}{3}\right)^2 = \frac{64}{27} - \frac{32}{9} = \frac{-32}{27} < 0.$$

- infinity: $w \to \infty$ as $x \to \infty$ and/or $y \to \infty$

(note: $y^3 - 2y^2 \to \infty$ as $y \to \infty$)

So the minimum is reached at $\boxed{(0, \tfrac{4}{3})}$.

Problem 5. (20)

a) (6) Find an equation of the tangent plane to the surface $\underbrace{x^4 + xyz}_{g(x,y,z)} = 7$ at $(1, 2, 3)$.

Normal vector: $\nabla g = \langle 4x^3 + yz, xz, xy \rangle = \langle 10, 3, 2 \rangle$.

Tangent plane: $\underline{10x + 3y + 2z = 22.}$

b) (6) Assume that the maximum value of $f(x, y, z)$ on the surface $x^4 + xyz = 7$ is attained at the point $(x, y, z) = (1, 2, 3)$. What can you say about ∇f at $(1, 2, 3)$?

By Lagrange, $\nabla f = \lambda \nabla g$ for some scalar λ

so $\nabla f = \langle 10\lambda, 3\lambda, 2\lambda \rangle$

(or: ∇f is parallel to $\langle 10, 3, 2 \rangle$)

c) (8) The relation $x^4 + xyz = 7$ implicitly defines x as a function of y and z, $x = x(y, z)$. Find the value of $\partial x/\partial y$ at the point $(x, y, z) = (1, 2, 3)$.

At $(1,2,3)$: differentiating $g(x,y,z) = 7$ gives

$$dg = 10\,dx + 3\,dy + 2\,dz = 0$$

Solve for dx: $dx = -\frac{3}{10}\,dy - \frac{2}{10}\,dz$.

Hence $\frac{\partial x}{\partial y} = \boxed{-\frac{3}{10}}$.