Math 53 Homework 9 – Solutions

15.6 # 44: $I_z = \iiint_E (x^2+y^2)\,\rho\,dV = \int_{-h}^{h}\int_{-\sqrt{h^2-x^2}}^{\sqrt{h^2-x^2}}\int_{\sqrt{x^2+y^2}}^{h}(x^2+y^2)\,k\,dz\,dy\,dx.$

Or better, in cylindrical coordinates: $I_z = \int_0^{2\pi}\int_0^h\int_r^h r^2\,k\,dz\,r\,dr\,d\theta.$

Inner: $[r^2kz]_r^h = khr^2 - kr^3$. Middle: $\int_0^h (khr^3 - kr^4)\,dr = \left[\frac{1}{4}khr^4 - \frac{1}{5}kr^5\right]_0^h = \frac{1}{20}kh^5.$

Outer: $2\pi\cdot\frac{1}{20}kh^5 = \frac{\pi}{10}kh^5.$

15.6 # 52: Volume $= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2} dz\,dy\,dx = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} dz\,r\,dr\,d\theta.$

Inner: $1-r^2$. Middle: $\int_0^1 (r-r^3)\,dr = \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = \frac{1}{4}$. Outer: $V = 2\pi\cdot\frac{1}{4} = \pi/2.$

Average value: $\bar f = \frac{1}{\pi/2}\iiint_E (x^2z+y^2z)\,dV = \frac{2}{\pi}\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r^2z\,dz\,r\,dr\,d\theta.$

Inner: $\left[\frac{1}{2}r^2z^2\right]_0^{1-r^2} = \frac{1}{2}r^2(1-r^2)^2.$

Middle: $\int_0^1 (\frac{1}{2}r^3 - r^5 + \frac{1}{2}r^7)\,dr = \left[\frac{1}{8}r^4 - \frac{1}{6}r^6 + \frac{1}{16}r^8\right]_0^1 = \frac{1}{8} - \frac{1}{6} + \frac{1}{16} = \frac{1}{48}.$

Outer: $\bar f = \frac{2}{\pi}(2\pi)\frac{1}{48} = \frac{1}{12}.$

15.7 # 9: (a) $z = r^2$; (b) $r^2 = 2r\sin\theta$, or $r = 2\sin\theta$.

15.7 # 15: solid cone centered on the z-axis, with vertex at the origin; the top face is a disk of radius 4 in the plane $z=4$.

$V = \int_0^4\int_0^{2\pi}\int_r^4 r\,dz\,d\theta\,dr = \int_0^4 (2\pi)(4-r)\,r\,dr = 2\pi\left[2r^2 - \frac{1}{3}r^3\right]_0^4 = 2\pi(32 - \frac{64}{3}) = 64\pi/3.$

15.7 # 18: The paraboloid $z = 1-x^2-y^2$ intersects the xy-plane in the circle $x^2+y^2 = r^2 = 1$ or $r=1$, so in cylindrical coordinates E is given by: $0\le\theta\le\pi/2$, $0\le r\le 1$, $0\le z\le 1-r^2$. Thus $\iiint_E (x^3+xy^2)\,dV = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2}(r\cos\theta)r^2\,r\,dz\,dr\,d\theta.$

Inner: $\int_0^{1-r^2} r^4\cos\theta\,dz = r^4(1-r^2)\cos\theta$. Middle: $\left[\frac{1}{5}r^5 - \frac{1}{7}r^7\right]_0^1\cos\theta = \frac{2}{35}\cos\theta.$

Outer: $\int_0^{\pi/2}\frac{2}{35}\cos\theta\,d\theta = \frac{2}{35}\left[\sin\theta\right]_0^{\pi/2} = \frac{2}{35}.$

15.7 # 22: E is the solid region within the cylinder $r=1$ bounded above and below by the sphere $r^2+z^2=4$. So its volume is $\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta = 2\pi\int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi\left[-\frac{2}{3}(4-r^2)^{3/2}\right]_0^1 = \frac{4}{3}\pi(8-3\sqrt{3}).$

15.8 # 9: (a) $z^2 = x^2+y^2$ (right angled cone centered on z-axis with vertex at origin) corresponds to $\phi = \pi/4$ and also $\phi = 3\pi/4$ (since the equation $z = \pm\sqrt{x^2+y^2}$ actually describes two cones, one centered on the positive z-axis and the other one centered on the negative z-axis). Or: $z^2 = x^2+y^2 \Leftrightarrow z^2 = r^2 \Leftrightarrow \rho^2\cos^2\phi = \rho^2\sin^2\phi \Leftrightarrow \cos^2\phi = \sin^2\phi$ ($\Leftrightarrow \cos\phi = \pm\sin\phi \Leftrightarrow \phi = \pi/4$ or $3\pi/4$.)

(b) $x^2+z^2 = 9 \Leftrightarrow (\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi)^2 = 9 \Leftrightarrow \rho^2(\sin^2\phi\cos^2\theta + \cos^2\phi) = 9.$

15.8 # 14: $\rho\le\csc\phi \Leftrightarrow \rho\sin\phi\le 1 \Leftrightarrow r\le 1$, or equivalently $x^2+y^2\le 1$, which corresponds to the solid cylinder of unit radius centered on the z-axis. Moreover, $\rho\le 2$ corresponds to the solid sphere of radius 2 centered at the origin. Hence, this solid is the intersection of the sphere and the cylinder.

(The sphere and the cylinder intersect at the two circles $r=1$, $z=\pm\sqrt{3}$; so the boundary of the solid is given by the portion of the cylinder where $-\sqrt{3}\le z\le\sqrt{3}$, and spherical caps at the top and bottom).

15.8 # 15: The cone $z = \sqrt{x^2+y^2}$ corresponds to $z = r$, i.e. $\rho\cos\phi = \rho\sin\phi$, i.e. $\phi = \pi/4$. (See also 15.8 # 9(a)). Thus, the region above the cone corresponds to $\phi \le \pi/4$.

In spherical coordinates, the sphere $x^2+y^2+z^2 = z$ (centered at $(0,0,\frac{1}{2})$ and of radius $\frac{1}{2}$, since the equation rewrites as $x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$) has equation $\rho = \cos\phi$. (This can be seen either geometrically on a slice by a vertical plane, or by manipulating the equation: $x^2+y^2+z^2 = z$ becomes $\rho^2 = \rho\cos\phi$, which simplifies to $\rho = \cos\phi$).

Hence, the solid is described by the inequalities $\rho \le \cos\phi$, $0 \le \phi \le \pi/4$.

(See Example 4 on page 1009 for a more detailed discussion and pictures).

15.8 # 19: In cylindrical coordinates: $0 \le z \le 2$, $0 \le r \le 2$, $0 \le \theta \le \pi/2$, so the integral is given by $\int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$.

In spherical coordinates: the top plane has equation $z = \rho\cos\phi = 2$, i.e. $\rho = 2\sec\phi$. The cylinder corresponds to $r = \rho\sin\phi = 3$, i.e. $\rho = 3\csc\phi$. They intersect when $2\sec\phi = 3\csc\phi$, i.e. $\tan\phi = 3/2$. Therefore:

$$\int_0^{\pi/2}\int_0^{\tan^{-1}(3/2)}\int_0^{2\sec\phi} f\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{\pi/2}\int_{\tan^{-1}(3/2)}^{\pi/2}\int_0^{3\csc\phi} f\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

15.8 # 23: The spheres correspond to $\rho = 1$ and $\rho = 2$, and the first octant corresponds to $0 \le \theta \le \pi/2$, $\phi \le \pi/2$. So $\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Inner: $\left[\frac{1}{4}\rho^4\cos\phi\sin\phi\right]_1^2 = \frac{15}{4}\cos\phi\sin\phi$.

Middle: $\frac{15}{4}\int_0^{\pi/2}\sin\phi\cos\phi, d\phi = \frac{15}{4}\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2} = \frac{15}{8}$.

Outer: $\frac{\pi}{2}\cdot\frac{15}{8} = \frac{15\pi}{16}$.

15.8 # 30: In spherical coordinates, the region below the cone $z = \sqrt{x^2+y^2}$ and above the xy-plane corresponds to $\pi/4 \le \phi \le \pi/2$. Therefore $\iiint_E dV = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Inner: $\left[\frac{1}{3}\rho^3\sin\phi\right]_0^2 = \frac{8}{3}\sin\phi$. Middle: $\left[-\frac{8}{3}\cos\phi\right]_{\pi/2}^{\pi/4} = \frac{8}{3}\frac{1}{\sqrt{2}} = \frac{4\sqrt{2}}{3}$. Outer: $2\pi\cdot\frac{4\sqrt{2}}{3} = \frac{8\pi\sqrt{2}}{3}$.

15.8 # 33: We take the hemisphere to be the region lying above the xy-plane and inside the sphere $x^2+y^2+z^2 = a^2$; and denote by K its (constant) density. So the base is contained in the xy-plane.

(a) By symmetry, the centroid lies on the z-axis, so we only need compute $\bar{z}$. Also, the mass of the hemisphere is $K\cdot(\text{volume}) = \frac{2}{3}K\pi a^3$. Therefore:

$\bar{z} = \frac{1}{\text{mass}}\iiint z\,K\,dV = \frac{3}{2K\pi a^3}\int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\phi)\,K\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Inner: $\left[\frac{1}{4}K\rho^4\cos\phi\sin\phi\right]_0^a = \frac{1}{4}Ka^4\cos\phi\sin\phi$.

Middle: $\frac{1}{4}Ka^4\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi = \frac{1}{4}Ka^4\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2} = \frac{1}{8}Ka^4$.

Outer: $\frac{3}{2K\pi a^3}(2\pi)\left(\frac{1}{8}Ka^4\right) = \frac{3}{8}a$. So the centroid is $(0,0,\frac{3}{8}a)$.

(b) We use the same setup as before, and compute the moment of inertia about the x-axis, I_x. (One could also compute I_y instead; by symmetry $I_x = I_y$).

$I_x = \iiint (y^2+z^2)\,K\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi)\,K\,\rho^2\sin\phi\,d\rho d\phi d\theta$.

Inner: $\frac{1}{5}Ka^5(\sin^3\phi\sin^2\theta + \cos^2\phi\sin\phi)$.

Middle: $\frac{1}{5}Ka^5\int_0^{\pi/2}\sin^2\theta(1-\cos^2\phi)\sin\phi + \cos^2\phi\sin\phi\,d\phi =$

$$= \frac{1}{5}Ka^5\left[\sin^2\theta(\tfrac{1}{3}\cos^3\phi - \cos\phi) - \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2} = \frac{1}{5}Ka^5(\tfrac{2}{3}\sin^2\theta + \tfrac{1}{3}).$$

Outer: $I_x = \frac{1}{5}Ka^5\int_0^{2\pi}(\frac{2}{3}\sin^2\theta + \frac{1}{3})\,d\theta = \frac{1}{5}Ka^5\int_0^{2\pi}(\frac{2}{3} - \frac{1}{3}\cos 2\theta)\,d\theta = \frac{4}{15}\pi Ka^5$.

Note: a more efficient setup for this calculation would have been to instead take the hemisphere to be the "right" half of the solid sphere $x^2 + y^2 + z^2 \le a^2$, i.e. where $y \ge 0$. The bounds are then $\rho \le a$, $0 \le \theta \le \pi$. Since the base is now a disk in the xz-plane, we can now compute the moment of inertia about the z-axis:
$I_z = \int_0^\pi \int_0^\pi \int_0^a (\rho^2 \sin^2\phi)\, K\, (\rho^2 \sin\phi)\, d\rho d\phi d\theta = \cdots = \frac{4}{15}\pi K a^5$.

15.8 # 35: In spherical coordinates $z = \sqrt{x^2+y^2}$ becomes $\phi = \pi/4$. So the volume is $V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2 \sin\phi\, d\rho d\phi d\theta = (\int_0^{2\pi} d\theta)(\int_0^{\pi/4} \sin\phi\, d\phi)(\int_0^1 \rho^2 d\rho) = \frac{2\pi}{3}[-\cos\phi]_0^{\pi/4} = \frac{\pi(2-\sqrt{2})}{3}$.
By symmetry the centroid is on the z-axis, i.e. $\bar{x} = \bar{y} = 0$, and $\bar{z} = \frac{1}{V}\iiint z\, dV$, so
$\bar{z} = \frac{3}{\pi(2-\sqrt{2})}\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\cos\phi\, \rho^2 \sin\phi\, d\rho d\phi d\theta = \frac{3}{\pi(2-\sqrt{2})}(2\pi)(\int_0^{\pi/4}\sin\phi\cos\phi\, d\phi)(\int_0^1 \rho^3\, d\rho)$
$\quad = \frac{3}{4(2-\sqrt{2})}\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/4} = \frac{3}{8(2-\sqrt{2})}$.

16.1 # 11: $\vec{F}(x,y) = \langle y, x\rangle$ corresponds to graph II. In the first quadrant all the vectors have positive x- and y-components, in the second quadrant they have positive x-components and negative y-components, etc. Moreover, the vectors get shorter as we approach the origin.

16.1 # 13: $\vec{F}(x,y) = \langle x-2, x+1\rangle$ corresponds to graph I since the vectors are independent of y (the vectors along vertical lines are identical) and, as we move to the right, both the x- and the y-components get larger.

16.1 # 18: $\vec{F}(x,y,z) = \langle x,y,z\rangle$ corresponds to graph II: each vector $\vec{F}(x,y,z)$ has the same length and direction as the position vector of the point (x,y,z), and therefore the vectors all point directly away from the origin.

16.1 # 26: $f(x,y) = \sqrt{x^2+y^2} \Rightarrow \nabla f = \langle f_x, f_y\rangle = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}\right\rangle = \dfrac{\langle x,y\rangle}{\sqrt{x^2+y^2}}$.
This vector has the same direction as $\langle x,y\rangle$ (= position vector of the point (x,y)), so points directly away from the origin; while its magnitude is $(x^2+y^2)^{-1/2}|\langle x,y\rangle| = 1$.
So $\nabla f(x,y)$ is the unit vector in the direction of $\langle x,y\rangle$ (directly away from the origin).

16.1 # 31: $f(x,y) = (x+y)^2 \Rightarrow \nabla f = \langle 2(x+y), 2(x+y)\rangle = 2(x+y)(\hat{\imath}+\hat{\jmath})$. So all the vectors are parallel to $\hat{\imath}+\hat{\jmath}$; they vanish on the line $x+y=0$ (or $y=-x$), and their magnitude increases with the distance to that line. This corresponds to plot II.

Problem 1: $M = \displaystyle\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} dxdydz = \int_0^1 (1-z^2)dz = 2/3.$

$$\bar{z} = \frac{1}{M}\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} z dxdydz = \frac{3}{2}\int_0^1 (z-z^3)dz = \frac{3}{8}.$$

$$\bar{x} = \frac{1}{M}\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} x dxdydz = \frac{3}{2}\int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1-z^2}{2} dydz$$
$$= \frac{3}{4}\int_0^1 (1-z^2)^{3/2}dz = \frac{3}{4}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{3}{4}\frac{3\pi}{16} = \frac{9\pi}{64} \qquad (z = \sin\theta,\ dz = \cos\theta\, d\theta)$$

using double angle formulas twice to calculate

$$\int_0^{\pi/2} \cos^4\theta\, d\theta = \int_0^{\pi/2} \tfrac{1}{4}(1+\cos 2\theta)^2\, d\theta = \int_0^{\pi/2} (\tfrac{3}{8} + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{8}\cos 4\theta)\, d\theta = \tfrac{3\pi}{16}.$$

By symmetry with respect to the plane $x = y$, $\bar{x} = \bar{y}$. Thus the centroid is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{9\pi}{64}, \frac{9\pi}{64}, \frac{3}{8} \right).$$

Problem 2: In cylindrical coordinates, distance to the origin is $d = \sqrt{r^2 + z^2}$, and

$$\begin{aligned}
\bar{d} &= \frac{1}{4\pi a^3/3} \int_0^{2a} \int_0^{2\pi} \int_0^{\sqrt{a^2-(z-a)^2}} \sqrt{r^2 + z^2}\, r\, dr d\theta dz \\
&= \frac{3}{4\pi a^3} \int_0^{2a} \int_0^{2\pi} \frac{1}{3}(r^2 + z^2)^{3/2} \Big|_{r=0}^{r=\sqrt{a^2-(z-a)^2}} d\theta dz \\
&= \frac{1}{4\pi a^3} \int_0^{2a} \int_0^{2\pi} ((2az)^{3/2} - z^3)\, d\theta dz \\
&= \frac{1}{2a^3} \int_0^{2a} ((2az)^{3/2} - z^3)\, dz \\
&= \frac{1}{2a^3} \left(\frac{2}{5}(2a)^4 - \frac{1}{4}(2a)^4 \right) = \frac{6}{5} a.
\end{aligned}$$

In spherical coordinates, $d = \rho$, $dV = \rho^2 \sin\phi\, d\rho d\phi d\theta$,

$$\begin{aligned}
\bar{d} &= \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2a\cos\phi} \rho\, \rho^2 \sin\phi\, d\rho d\phi d\theta \\
&= \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{4}(2a\cos\phi)^4 \sin\phi\, d\phi d\theta \\
&= \frac{3}{4\pi a^3} \int_0^{2\pi} \frac{1}{4}(2a)^4 \frac{-1}{5}(\cos\phi)^5 \Big|_0^{\pi/2} d\theta \\
&= \frac{3}{4\pi a^3} \frac{8\pi a^4}{5} = \frac{6a}{5}.
\end{aligned}$$