Math 53 Homework 8 – Solutions

15.4 # 36: a) Using polar coordinates: $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta =$
$= 2\pi\int_0^\infty re^{-r^2}\,dr = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty = 2\pi(0+\frac{1}{2}) = \pi.$
Or, to be more rigorous, we integrate over the disk D_a and take the limit as $a\to\infty$:

$$\iint_{D_a} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi(1-e^{-a^2}).$$

The result then follows, since $\lim_{a\to\infty}\pi(1-e^{-a^2}) = \pi$.
b) $\iint_{S_a} e^{-(x^2+y^2)}dA = \int_{-a}^a\int_{-a}^a e^{-x^2}e^{-y^2}\,dy\,dx = \left(\int_{-a}^a e^{-x^2}dx\right)\left(\int_{-a}^a e^{-y^2}dy\right).$
Taking the limit as $a\to\infty$ on both sides, we deduce:

$$\pi = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}dA = \left(\int_{-\infty}^\infty e^{-x^2}dx\right)\left(\int_{-\infty}^\infty e^{-y^2}dy\right).$$

c) Since $\int_{-\infty}^\infty e^{-y^2}dy = \int_{-\infty}^\infty e^{-x^2}dx$ (the name of the integration variable is irrelevant), the result of (b) becomes: $\left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)^2 = \pi$. Taking the square root (and observing that $e^{-x^2}>0$ for all x so the integral is positive), we get: $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$.
d) Letting $x = t/\sqrt{2}$, we have $\int_{-\infty}^\infty e^{-x^2}dx = \int_{-\infty}^\infty \frac{1}{\sqrt{2}}e^{-t^2/2}\,dt$. Hence, $\sqrt{\pi} = \frac{1}{\sqrt{2}}\int_{-\infty}^\infty e^{-t^2/2}\,dt$. Equivalently, $\int_{-\infty}^\infty e^{-t^2/2}\,dt = \sqrt{2\pi}$.

15.5 # 8: $m = \iint_D \rho\,dA = \int_0^1\int_0^{\sqrt{x}} x\,dy\,dx = \int_0^1 [xy]_{y=0}^{y=\sqrt{x}}\,dx = \int_0^1 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_0^1 = \frac{2}{5}.$
$\bar{x} = \frac{1}{m}\iint_D x\rho\,dA = \frac{5}{2}\int_0^1\int_0^{\sqrt{x}} x^2\,dy\,dx = \frac{5}{2}\int_0^1 [x^2y]_{y=0}^{y=\sqrt{x}}\,dx = \frac{5}{2}\int_0^1 x^{5/2}\,dx = \frac{5}{2}\left[\frac{2}{7}x^{7/2}\right]_0^1 = \frac{5}{7}.$
$\bar{y} = \frac{1}{m}\iint_D y\rho\,dA = \frac{5}{2}\int_0^1\int_0^{\sqrt{x}} xy\,dy\,dx = \frac{5}{2}\int_0^1 [\frac{1}{2}xy^2]_{y=0}^{y=\sqrt{x}}\,dx = \frac{5}{2}\int_0^1 \frac{1}{2}x^2\,dx = \frac{5}{2}\left[\frac{1}{6}x^3\right]_0^1 = \frac{5}{12}.$

15.5 # 12: $\rho(x,y) = k(x^2+y^2) = kr^2$, $m = \int_0^{\pi/2}\int_0^1 kr^2\,r\,dr\,d\theta = \frac{\pi}{2}\int_0^1 kr^3\,dr = \frac{\pi}{8}k.$
$\bar{x} = \frac{1}{m}\iint x\,\rho\,dA = \frac{1}{m}\int_0^{\pi/2}\int_0^1 (r\cos\theta)\,(kr^2)\,r\,dr\,d\theta = \frac{1}{m}\int_0^{\pi/2}\left[\frac{1}{5}kr^5\cos\theta\right]_{r=0}^{r=1}\,d\theta =$
$= \frac{1}{m}\frac{k}{5}\int_0^{\pi/2}\cos\theta\,d\theta = \frac{k/5}{k\pi/8}\left[\sin\theta\right]_0^{\pi/2} = \frac{8}{5\pi}.$
$\bar{y} = \frac{1}{m}\iint y\,\rho\,dA = \frac{1}{m}\int_0^{\pi/2}\int_0^1 (r\sin\theta)\,(kr^2)\,r\,dr\,d\theta = \frac{1}{m}\int_0^{\pi/2}\left[\frac{1}{5}kr^5\sin\theta\right]_{r=0}^{r=1}\,d\theta =$
$= \frac{1}{m}\frac{k}{5}\int_0^{\pi/2}\sin\theta\,d\theta = \frac{k/5}{k\pi/8}\left[-\cos\theta\right]_0^{\pi/2} = \frac{8}{5\pi}.$
(Note: $\bar{x} = \bar{y}$ by symmetry: the lamina is symmetric about the axis $y = x$, and so is the density, so the center of mass lies on the symmetry axis.)

15.5 # 18: $I_x = \iint_D y^2\,\rho\,dA = \int_0^{\pi/2}\int_0^1 (r^2\sin^2\theta)(kr^2)\,r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{1}{6}kr^6\sin^2\theta\right]_0^1\,d\theta =$
$= \frac{1}{6}k\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{6}k\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{6}k\left[\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{24}k.$
$I_y = \iint_D x^2\,\rho\,dA = \int_0^{\pi/2}\int_0^1 (r^2\cos^2\theta)(kr^2)\,r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{1}{6}kr^6\cos^2\theta\right]_0^1\,d\theta =$
$= \frac{1}{6}k\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{6}k\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{1}{6}k\left[\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{24}k.$
$I_0 = \iint_D r^2\,\rho\,dA = \int_0^{\pi/2}\int_0^1 r^2(kr^2)\,r\,dr\,d\theta = \frac{\pi}{2}\left[\frac{1}{6}kr^6\right]_0^1 = \frac{\pi}{12}k.$
(Note: by symmetry, $I_y = I_x$; and as a general fact, $I_0 = I_x + I_y$; so it was enough to compute one of the three moments of inertia).

15.5 # 28: a) $f(x,y)\geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here $f(x,y) = 0$ outside of the unit square, so we just need to compute $\int_0^1\int_0^1 f(x,y)\,dy\,dx =$

$\int_0^1\int_0^1 4xy\,dy\,dx = \int_0^1 \left[2xy^2\right]_{y=0}^{y=1} dx = \int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1.$

b) (i) The region where $x \geq \frac{1}{2}$ corresponds to the right half of the unit square (recall that X and Y only take values between 0 and 1).

So $P(X \geq \frac{1}{2}) = \int_{1/2}^1\int_0^1 4xy\,dy\,dx = \int_{1/2}^1 \left[2xy^2\right]_{y=0}^{y=1} dx = \int_{1/2}^1 2x\,dx = \left[x^2\right]_{1/2}^1 = \frac{3}{4}.$

(ii) $P(X \geq \frac{1}{2},\, Y \leq \frac{1}{2}) = \int_{1/2}^1\int_0^{1/2} 4xy\,dy\,dx = \int_{1/2}^1 \left[2xy^2\right]_0^{1/2} dx = \int_{1/2}^1 \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_{1/2}^1 = \frac{3}{16}.$

c) $E(X) = \iint_{\mathbb{R}^2} x\,f(x,y)\,dA = \int_0^1\int_0^1 x(4xy)\,dy\,dx = \int_0^1 [2x^2y^2]_0^1\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}.$

$E(Y) = \iint_{\mathbb{R}^2} y\,f(x,y)\,dA = \int_0^1\int_0^1 y(4xy)\,dy\,dx = \int_0^1 [\frac{4}{3}xy^3]_0^1\,dx = \int_0^1 \frac{4}{3}x\,dx = \frac{2}{3}.$

15.9 # 3: $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ -e^{-r}\cos\theta & -e^{-r}\sin\theta \end{vmatrix} = e^{-2r}(\sin^2\theta + \cos^2\theta) = e^{-2r}.$

15.9 # 11: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, and $x - 3y = (2u+v) - 3(u+2v) = -u - 5v.$

To find the region S in the uv-plane that corresponds to R, we first find how the boundary maps under the given transformation. The line through (0,0) and (2,1) is $y = \frac{1}{2}x$; so it corresponds to $u + 2v = \frac{1}{2}(2u+v)$, which simplifies to $v = 0$. The line though (0,0) and (1,2) is $y = 2x$; this corresponds to $u + 2v = 2(2u+v)$, which simplifies to $u = 0$. Finally, the line through (1,2) and (2,1) is $x + y = 3$, which becomes $(2u+v) + (u+2v) = 3$, which simplifies to $u + v = 1$. So S is the triangle in the uv-plane bounded by the lines $u = 0$, $v = 0$, and $u + v = 1$ (i.e. $v = 1 - u$).

Therefore $\iint_R (x - 3y)\,dA = \int_0^1\int_0^{1-u}(-u-5v)\,|3|\,dv\,du = -3\int_0^1\int_0^{1-u}(u+5v)\,dv\,du.$

Inner: $\int_0^{1-u}(u+5v)\,dv = \left[uv + \frac{5}{2}v^2\right]_0^{1-u} = u(1-u) + \frac{5}{2}(1-u)^2 = \frac{3}{2}u^2 - 4u + \frac{5}{2}.$

Outer: $-3\int_0^1(\frac{3}{2}u^2 - 4u + \frac{5}{2})\,du = -3\left[\frac{1}{2}u^3 - 2u^2 + \frac{5}{2}u\right]_0^1 = -3.$

15.9 # 15: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$. The integrand is $xy = u$.

Since $xy = u$, the hyperbolas $xy = 1$ and $xy = 3$ correspond to the lines $u = 1$ and $u = 3$ respectively. Moreover $y = x \Leftrightarrow v = \frac{u}{v} \Leftrightarrow v^2 = u$, and $y = 3x \Leftrightarrow v = 3\frac{u}{v} \Leftrightarrow v^2 = 3u$.

Since we are in the first quadrant, $y \geq 0$ so $v \geq 0$. Hence the region of integration corresponds to $\sqrt{u} \leq v \leq \sqrt{3u}$, $1 \leq u \leq 3$. Thus

$$\iint_R xy\,dA = \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} u\,\frac{1}{v}\,dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right)\,du = \int_1^3 u\ln\sqrt{3}\,du =$$

$$= \left[\tfrac{1}{2}u^2\right]_1^3 \ln\sqrt{3} = 4\ln\sqrt{3} = 2\ln 3.$$

15.9 # 19: Let $u = x - 2y$ and $v = 3x - y$: then $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5$, so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{5}.$

With this change of variables, R is the image of the rectangle $0 \leq u \leq 4$, $1 \leq v \leq 8$. So

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4\int_1^8 \frac{u}{v}\,\frac{1}{5}\,dv\,du = \frac{1}{5}\left(\int_0^4 u\,du\right)\left(\int_1^8 \frac{dv}{v}\right) = \frac{1}{5}\left[\frac{u^2}{2}\right]_0^4 \Big[\ln v\Big]_1^8 = \frac{8}{5}\ln 8.$$

15.6 # 9: $\iiint_E 2x\,dV = \int_0^2\int_0^{\sqrt{4-y^2}}\int_0^y 2x\,dz\,dx\,dy$. Inner: $[2xz]_0^y = 2xy.$

Middle: $\int_0^{\sqrt{4-y^2}} 2xy\,dx = \left[x^2y\right]_0^{\sqrt{4-y^2}} = (4-y^2)y$. Outer: $\int_0^2 4y - y^3\,dy = \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 4.$

15.6 # 15: The base of the tetrahedron is the triangle with vertices (0,0),(1,0),(0,1) in the xy-plane, i.e. the region $0 \le y \le 1-x$, $0 \le x \le 1$. The top face is $x+y+z=1$, i.e. $z = 1-x-y$, while the bottom face is $z=0$. So $\iiint_T x^2\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2\,dz\,dy\,dx$.
Inner: $[x^2 z]_0^{1-x-y} = x^2(1-x-y)$.
Middle: $\int_0^{1-x} x^2(1-x-y)\,dy = x^2\left[(1-x)y - \frac{1}{2}y^2\right]_0^{1-x} = x^2((1-x)^2 - \frac{(1-x)^2}{2}) = \frac{1}{2}x^2(1-x)^2$.
Outer: $\int_0^1 \frac{1}{2}x^2(1-x)^2\,dx = \int_0^1 \frac{1}{2}x^2 - x^3 + \frac{1}{2}x^4\,dx = \left[\frac{1}{6}x^3 - \frac{1}{4}x^4 + \frac{1}{10}x^5\right]_0^1 = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{1}{60}$.

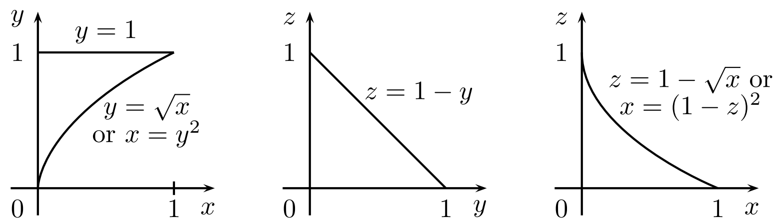
15.6 # 21: the solid is a piece of the cylinder of radius 3 centered on the z-axis, between the planes $z=1$ (bottom) and $z = 5-y$ (top).

$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} dz\,dy\,dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5-y-1)\,dy\,dx = \int_{-3}^{3} \left[4y - \tfrac{1}{2}y^2\right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} 8\sqrt{9-x^2}\,dx = \int_{-\pi/2}^{\pi/2} 8(3\cos t)\,(3\cos t)\,dt = 72\int_{-\pi/2}^{\pi/2} \cos^2 t\,dt = 36\pi.$$

(where we used the substitution $x = 3\sin t$, $dx = 3\cos t\,dt$). Or, in cylindrical coordinates:
$V = \int_0^{2\pi} \int_0^3 \int_1^{5-r\sin\theta} dz\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^3 (4 - r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi} \left[2r^2 - \frac{1}{3}r^3\sin\theta\right]_0^3 d\theta$
$= \int_0^{2\pi} (18 - 9\sin\theta)\,d\theta = 36\pi$.

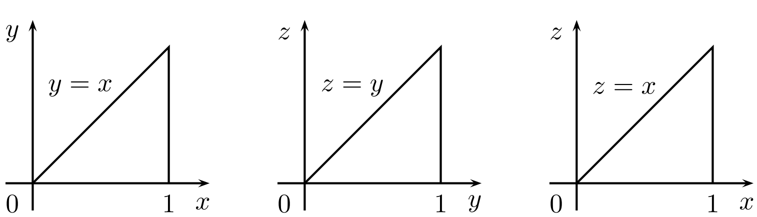
15.6 # 33: The projections of E on the coordinate planes are:

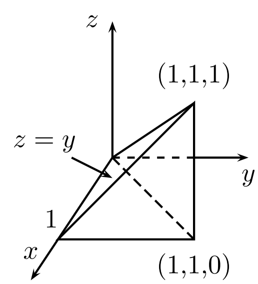


$0 \le z \le 1-y$ $\qquad$ $0 \le x \le y^2$ $\qquad$ $\sqrt{x} \le y \le 1-z$

$$\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx &= \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\,dz\,dx\,dy \\
&= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz \\
&= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy \\
&= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx \\
&= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz
\end{aligned}$$

15.6 # 35: The region and its projections are:





$0 \le z \le y$ $\qquad$ $y \le x \le 1$ $\qquad$ $z \le y \le x$

$$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dx\,dz\,dy$$
$$= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx = \int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz$$

Problem 1:

a) Average distance $= \dfrac{1}{\text{Area}} \iint r\, dA = \dfrac{1}{\pi a^2} \int_0^{2\pi} \int_0^a r\, r\, dr\, d\theta = \dfrac{1}{\pi a^2}\, 2\pi \left[\dfrac{1}{3} r^3\right]_0^a = \dfrac{2}{3} a.$

b) (Using the setup suggested by the hint, so the circle has polar equation $r = 2a\cos\theta$):

Average distance $= \dfrac{1}{\text{Area}} \iint_r dA = \dfrac{1}{\pi a^2} \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^2\, dr\, d\theta = \dfrac{1}{\pi a^2} \int_{-\pi/2}^{\pi/2} \dfrac{1}{3} (2a\cos\theta)^3\, d\theta$

$= \dfrac{8a^3}{3\pi a^2} \int_{-\pi/2}^{\pi/2} \cos^3\theta\, d\theta = \dfrac{8a}{3\pi} \int_{-\pi/2}^{\pi/2} (1 - \sin^2\theta) \cos\theta\, d\theta =$ (substituting $u = \sin\theta$)

$= \dfrac{8a}{3\pi} \int_{-1}^{1} (1 - u^2)\, du = \dfrac{8a}{3\pi} \left[u - \dfrac{1}{3} u^3\right]_{-1}^{1} = \dfrac{8a}{3\pi} (\dfrac{2}{3} - (-\dfrac{2}{3})) = \dfrac{32a}{9\pi}.$

Problem 2:

a) Area $= \displaystyle\int_0^{\pi/2} \int_0^{\sin 2\theta} r\, dr\, d\theta.$ Inner: $\left[\frac{1}{2} r^2\right]_0^{\sin 2\theta} = \frac{1}{2}\sin^2 2\theta.$

Outer: $\frac{1}{2} \displaystyle\int_0^{\pi/2} \frac{1}{2}(1 - \cos 4\theta)\, d\theta = \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta \Big|_0^{\pi/2} = \frac{\pi}{8}.$

b) By symmetry the centroid must be on the diagonal line $y = x$, so calculating $\bar{x}$ is enough.

$\bar{x} = \dfrac{1}{\text{Area}} \displaystyle\int_0^{\pi/2} \int_0^{\sin 2\theta} r\cos\theta\, r\, dr\, d\theta.$

Inner: $\frac{1}{3} r^3 \cos\theta \Big]_0^{\sin 2\theta} = \frac{1}{3}\sin^3 2\theta \cos\theta = \frac{1}{3}(2\sin\theta\cos\theta)^3 \cos\theta = \frac{8}{3}\sin^3\theta\cos^4\theta$

$= \frac{8}{3}\sin\theta\cos^4\theta(1 - \cos^2\theta) = \frac{8}{3}\sin\theta(\cos^4\theta - \cos^6\theta).$

Outer: $\frac{8}{3}\left(-\frac{1}{5}\cos^5\theta + \frac{1}{7}\cos^7\theta\right)\Big]_0^{\pi/2} = \frac{8}{3}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{16}{105}$. Therefore $\bar{x} = \bar{y} = \dfrac{8}{\pi}\dfrac{16}{105} = \dfrac{128}{105\pi}.$

Problem 3: $u = xy$, $v = y/x$: so $uv = y^2$ and $u/v = x^2$, which gives $x^2 + y^2 = uv + u/v$.

The Jacobian is $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \dfrac{2y}{x} = 2v.$

Thus $du\, dv = \left|\dfrac{2y}{x}\right| dx\, dy$, and $dx\, dy = \left|\dfrac{x}{2y}\right| du\, dv = \dfrac{1}{2|v|}\, du\, dv.$

Limits of integration: $0 \le xy \le 1$, $1 \le x \le 2$. In uv-coordinates, the first inequality becomes $0 \le u \le 1$; and the second one becomes $1 \le x^2 = u/v \le 4$, or equivalently $v \le u \le 4v$, which means that $v \le u$ and $v \ge \frac{1}{4}u$. So

$$\begin{aligned}
\iint_R (x^2 + y^2)\, dx\, dy &= \int_0^1 \int_{u/4}^{u} \left(uv + \frac{u}{v}\right) \frac{1}{2v}\, dv\, du \\
&= \int_0^1 \int_{u/4}^{u} \left(\frac{u}{2} + \frac{u}{2v^2}\right) dv\, du \\
&= \int_0^1 \left[\frac{uv}{2} - \frac{u}{2v}\right]_{u/4}^{u} du \\
&= \int_0^1 \left(\left(\frac{1}{2}u^2 - \frac{1}{2}\right) - \left(\frac{1}{8}u^2 - 2\right)\right) du \\
&= \int_0^1 \left(\frac{3}{8}u^2 + \frac{3}{2}\right) du = \left[\frac{1}{8}u^3 + \frac{3}{2}u\right]_0^1 = \frac{1}{8} + \frac{3}{2} = \frac{13}{8}.
\end{aligned}$$