Math 53 Homework 7 – Solutions

Problem 1. a) The variables are m and b, and we are trying to minimize $f(m,b) = \sum_{i=1}^{n} (y_i - (mx_i + b))^2$. Critical points are solutions of the two equations

$f_m = \sum_{i=1}^{n} -2x_i(y_i - (mx_i + b)) = 0$, so $\sum_{i=1}^{n}(-x_iy_i + mx_i^2 + bx_i) = 0$, or equivalently,

$$m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_iy_i;$$

and $f_b = \sum_{i=1}^{n} -2(y_i - (mx_i + b)) = 0$, so $\sum_{i=1}^{n}(-y_i + mx_i + b) = 0$, or equivalently,

$$m\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i.$$

Thus the critical points are the solutions of the given equations. (Observe: given the data x_i and y_i, this is merely a 2×2 linear system!) We will not prove here that the critical point is a minimum.

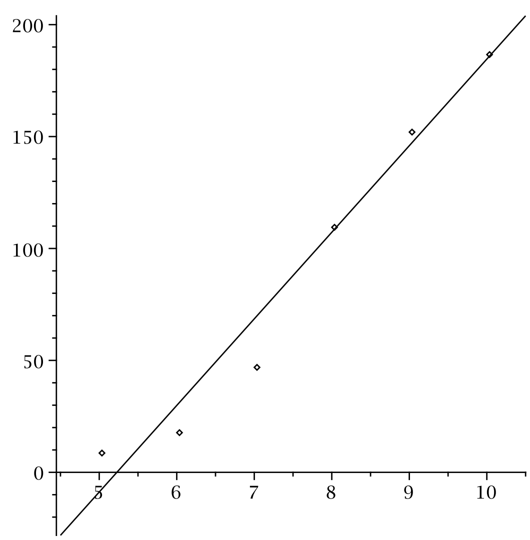
b) We calculate $\sum x_i = 45$, $\sum x_i^2 = 355$, $\sum y_i = 527.4$, and $\sum x_iy_i = 4632.4$. So the equations are

$$\begin{cases} 45m + \ \ 6b = 527.4 \\ 355m + 45b = 4632.4 \end{cases}$$

This gives $m = 38.68$ and $b = -202.2$.

c) We calculate the predicted values (and plot the data as well):

x	5	7	9
Predicted $(mx+b)$	-8.8	68.6	145.9
Actual	9.8	48.1	152.7



For $x = 5$ the predicted value is clearly invalid. In fact the historical data from 2000 to 2008 is much better approximated by an *exponential* function $y = \exp(m'x + b')$, so over that range of years one should try to find the best fit line for $\ln(y_i)$ vs. x_i instead; whereas the more recent data is reasonably well approximated by a linear function.

d) According to $y = mx + b$, about 223.3 billion text messages will be sent in December 2011 ($x = 11$), and about 571.4 billion will be sent in December 2020.

Problem 2. a) $w = x^2 + y^2 + z^2$ where $y^2 + xz = 2$:

(i) $f(x, y, z) = x^2 + y^2 + z^2$, and $\partial f/\partial x = 2x$. In this case y and z are being held constant (but $y^2 + xz$ is not being held constant).

(ii) we use the relation $y^2 + xz = 2$ to solve for z: namely, $z = (2 - y^2)/x$. Therefore $w = g(x, y) = x^2 + y^2 + \dfrac{(2 - y^2)^2}{x^2}$. This gives $\left(\dfrac{\partial w}{\partial x}\right)_y = \dfrac{\partial g}{\partial x} = 2x - \dfrac{2(2 - y^2)^2}{x^3}$.

Here we differentiate with respect to x while y is being held constant, but z varies (so that the relation $y^2 + xz = 2$ remains valid).

(iii) we use the relation $y^2 + xz = 2$ to eliminate y: namely, $y^2 = 2 - xz$, so $w = h(x, z) = x^2 + (2 - xz) + z^2$. This gives $\left(\dfrac{\partial w}{\partial x}\right)_z = \dfrac{\partial h}{\partial x} = 2x - z$.

Here we differentiate with respect to x while z is being held constant, but y varies (so that the relation $y^2 + xz = 2$ remains valid).

(To see that these partial derivatives are really different, observe that for $(x, y, z) = (1, 1, 1)$, we get $\partial f/\partial x = 2$, but $\partial g/\partial x = 0$ and $\partial h/\partial x = 1$! This is of course because these quantities correspond to scenarios where different quantities are being held constant.)

b) Differentiating $w = x^2 + y^2 + z^2$, we get: $dw = 2x\,dx + 2y\,dy + 2z\,dz$. Differentiating the constraint equation $y^2 + xz = 2$, we get: $z\,dx + 2y\,dy + x\,dz = 0$.

Treating x, y as independent variables: $z\,dx + 2y\,dy + x\,dz = 0 \Rightarrow dz = -\dfrac{z}{x}\,dx - \dfrac{2y}{x}\,dy$.

Substituting into dw, we get:

$$dw = 2x\,dx + 2y\,dy + 2z\left(-\frac{z}{x}\,dx - \frac{2y}{x}\,dy\right) = \left(2x - \frac{2z^2}{x}\right)dx + \left(2y - \frac{4yz}{x}\right)dy.$$

So $\left(\dfrac{\partial w}{\partial x}\right)_y = 2x - \dfrac{2z^2}{x}$ (which is consistent with the formula in (a)).

Next, viewing x, z as independent: $z\,dx + 2y\,dy + x\,dz = 0 \Rightarrow dy = -\dfrac{z}{2y}\,dx - \dfrac{x}{2y}\,dz$.

Substituting into dw:

$$dw = 2x\,dx + 2y\left(-\frac{z}{2y}\,dx - \frac{x}{2y}\,dz\right) + 2z\,dz = (2x - z)\,dx + (2z - x)\,dz.$$

So $\left(\dfrac{\partial w}{\partial x}\right)_z = 2x - z$.

15.2 # 12: $\displaystyle\int_0^1 xy\sqrt{x^2 + y^2}\,dy = \left[\tfrac{1}{3}x(x^2 + y^2)^{3/2}\right]_0^1 = \tfrac{1}{3}x(x^2 + 1)^{3/2} - \tfrac{1}{3}x^4$.

So $\displaystyle\int_0^1\int_0^1 xy\sqrt{x^2 + y^2}\,dy\,dx = \frac{1}{3}\int_0^1 x(x^2 + 1)^{3/2} - x^4\,dx =$

$$= \frac{1}{15}\left[(x^2 + 1)^{5/2} - x^5\right]_0^1 = \frac{1}{15}\left[(2^{5/2} - 1) - (1 - 0)\right] = \frac{4\sqrt{2} - 2}{15}.$$

15.2 # 17: $\displaystyle\iint_R \frac{xy^2}{x^2 + 1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2 + 1}\,dy\,dx.$

Inner: $\displaystyle\left[\frac{x}{x^2 + 1}\,\frac{y^3}{3}\right]_{-3}^3 = \frac{x}{x^2 + 1}\left(\frac{27 - (-27)}{3}\right) = 18\frac{x}{x^2 + 1}.$

Outer: $\displaystyle\int_0^1 18\frac{x}{x^2+1}\,dx = 18\left[\frac{1}{2}\ln(x^2+1)\right]_0^1 = 9\ln 2.$

15.2 # 21: $\iint_R xye^{x^2y}\,dA = \int_0^2\int_0^1 xye^{x^2y}\,dx\,dy = \int_0^2\left[\frac{1}{2}e^{x^2y}\right]_{x=0}^{x=1}dy = \frac{1}{2}\int_0^2(e^y-1)\,dy$

$$= \tfrac{1}{2}\left[e^y-y\right]_0^2 = \tfrac{1}{2}((e^2-2)-(e^0-0)) = \tfrac{1}{2}(e^2-3).$$

(Note: integrating in the other order is quite a bit harder.)

15.3 # 3: $\int_0^1\int_{x^2}^x(1+2y)\,dy\,dx = \int_0^1[y+y^2]_{y=x^2}^{y=x}\,dx = \int_0^1((x+x^2)-(x^2+x^4))\,dx$

$$= \int_0^1(x-x^4)\,dx = \left[\tfrac{1}{2}x^2-\tfrac{1}{5}x^5\right]_0^1 = \tfrac{1}{2}-\tfrac{1}{5} = \tfrac{3}{10}.$$

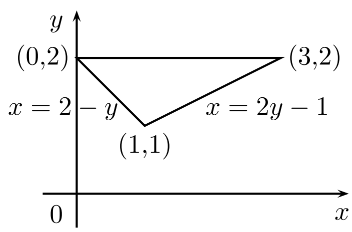
15.3 # 6: $\int_0^1\int_0^v\sqrt{1-v^2}\,du\,dv = \int_0^1\left[u\sqrt{1-v^2}\right]_{u=0}^{u=v}dv = \int_0^1 v\sqrt{1-v^2}\,dv$

$$= -\tfrac{1}{3}(1-v^2)^{3/2}\Big]_0^1 = \tfrac{1}{3}.$$

15.3 # 8: $\displaystyle\iint_D\frac{y}{x^5+1}\,dA = \int_0^1\int_0^{x^2}\frac{y}{x^5+1}\,dy\,dx = \int_0^1\left[\frac{y^2/2}{x^5+1}\right]_0^{x^2}dx = \int_0^1\frac{1}{2}\frac{x^4}{x^5+1}\,dx$

$$= \tfrac{1}{10}\ln(x^5+1)\Big]_0^1 = \tfrac{1}{10}\ln 2.$$

15.3 # 15:



$$\int_1^2\int_{2-y}^{2y-1} y^3\,dx\,dy$$

(or also: $\displaystyle\int_0^1\int_{2-x}^2 y^3\,dy\,dx + \int_1^3\int_{\frac{x+1}{2}}^2 y^3\,dy\,dx$)

Inner: $\int_{2-y}^{2y-1} y^3\,dx = \left[y^3x\right]_{x=2-y}^{x=2y-1} = y^3((2y-1)-(2-y)) = 3y^4-3y^3.$

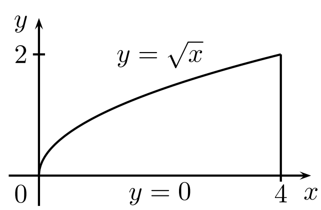
Outer: $\int_1^2 3y^4-3y^3\,dy = \left[\frac{3}{5}y^5-\frac{3}{4}y^4\right]_1^2 = \frac{96}{5}-12-\frac{3}{5}+\frac{3}{4} = \frac{147}{20}.$

15.3 # 23: The plane $3x+2y+z=6$ (or $z=6-3x-2y$) intersects the xy-plane ($z=0$) along the line $3x+2y=6$ (with x-intercept 2 and y-intercept 3), or $y=3-\frac{3}{2}x$. So the region of integration is the triangle with vertices (0,0), (2,0) and (0,3): $x\ge 0$, $0\le y\le 3-\frac{3}{2}x$, and the integrand is $6-3x-2y$.

$$\int_0^2\int_0^{3-\frac{3}{2}x}(6-3x-2y)\,dy\,dx = \int_0^2\left[6y-3xy-y^2\right]_0^{3-\frac{3}{2}x}dx = \int_0^2\left[(6-3x-y)y\right]_0^{3-\frac{3}{2}x}dx =$$

$$= \int_0^2(3-\tfrac{3}{2}x)^2\,dx = -\tfrac{2}{9}(3-\tfrac{3}{2}x)^3\Big|_0^2 = \tfrac{2}{9}3^3 = 6.$$

15.3 # 39:

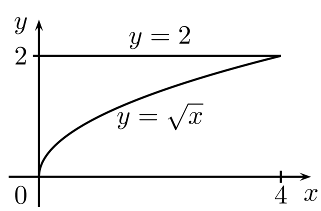


Region of integration: $0\le y\le\sqrt{x}$, $0\le x\le 4$

or equivalently: $y^2\le x\le 4$, $0\le y\le 2$

So: $\int_0^4\int_0^{\sqrt{x}} f(x,y)\,dy\,dx = \int_0^2\int_{y^2}^4 f(x,y)\,dx\,dy.$

15.3 # 47:



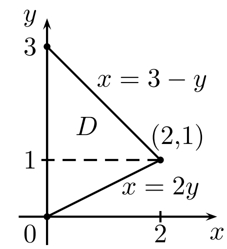
Region of integration: $\sqrt{x}\le y\le 2$, $0\le x\le 4$

or equivalently: $0\le y^2\le x$, $0\le y\le 2$

So $\displaystyle\int_0^4\int_{\sqrt{x}}^2\frac{1}{y^3+1}\,dy\,dx = \int_0^2\int_0^{y^2}\frac{1}{y^3+1}\,dx\,dy.$

Inner: $\left[\dfrac{x}{y^3+1}\right]_0^{y^2} = \dfrac{y^2}{y^3+1}$. Outer: $\displaystyle\int_0^2 \frac{y^2}{y^3+1}\,dy = \tfrac{1}{3}\ln(y^3+1)\big|_0^2 = \frac{1}{3}\ln 9.$

15.3 # 58:



$$\int_0^1\int_0^{2y} f(x,y)\,dx\,dy + \int_1^3\int_0^{3-y} f(x,y)\,dx\,dy$$
$$= \iint_D f(x,y)\,dA = \int_0^2\int_{x/2}^{3-x} f(x,y)\,dy\,dx.$$

15.4 # 6: $\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$ represents the area of the region $0 \le r \le 4\cos\theta$, $0 \le \theta \le \pi/2$. Since $r = 4\cos\theta$ is the polar equation of the circle of radius 2 with center at $(2,0)$, R is the upper half of the disk enclosed by this circle; and the area is 2π. Evaluating the integral:

Inner: $\left[\frac{1}{2}r^2\right]_0^{4\cos\theta} = 8\cos^2\theta$.

Outer: $\int_0^{\pi/2} 8\cos^2\theta\,d\theta = \int_0^{\pi/2} 4(1+\cos 2\theta)\,d\theta = [4\theta + 2\sin 2\theta]_0^{\pi/2} = 2\pi$.

15.4 # 7: $\iint_D xy\,dA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$.

Inner: $\int_0^3 r^3\sin\theta\cos\theta\,dr = \sin\theta\cos\theta\left[\frac{1}{4}r^4\right]_0^3 = \frac{3^4}{4}\sin\theta\cos\theta$.

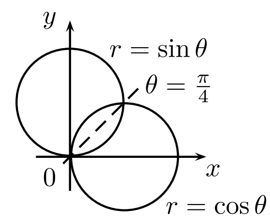
Outer: $\frac{3^4}{4}\int_0^{2\pi}\sin\theta\cos\theta\,d\theta = \frac{3^4}{4}\left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0$. (This was obvious by symmetry...)

15.4 # 17: By symmetry, $A = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta$.

Inner: $\left[\frac{1}{2}r^2\right]_0^{\sin\theta} = \frac{1}{2}\sin^2\theta$.

Outer: $\int_0^{\pi/4}\sin^2\theta\,d\theta = \int_0^{\pi/4}\frac{1}{2}(1-\cos 2\theta)\,d\theta =$

$\qquad = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) = \frac{\pi}{8} - \frac{1}{4}$.



15.4 # 25: The cone $z = \sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2 = 1$ when $x^2+y^2+(\sqrt{x^2+y^2})^2 = 1$, or $x^2+y^2 = \frac{1}{2}$. So

$$V = \iint_{x^2+y^2\le 1/2}\left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\left(\sqrt{1-r^2} - r\right) r\,dr\,d\theta =$$

$= 2\pi\int_0^{1/\sqrt{2}}(r\sqrt{1-r^2} - r^2)\,dr = 2\pi\left[-\frac{1}{3}(1-r^2)3/2 - \frac{1}{3}r^3\right]_0^{1/\sqrt{2}} = 2\pi(-\frac{1}{3})(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - 1)$

$= \frac{\pi}{3}(2-\sqrt{2})$.

15.4 # 31: $\int_0^1\int_y^{\sqrt{2-y^2}}(x+y)\,dx\,dy$: the region of integration is the portion of the disk $x^2+y^2 \le 2$ where $x \ge y$ and $y \ge 0$, i.e. between the x-axis and the line $y = x$.

So $\int_0^1\int_y^{\sqrt{2-y^2}}(x+y)\,dx\,dy = \int_0^{\pi/4}\int_0^{\sqrt{2}}(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$.

Inner: $\frac{1}{3}r^3(\cos\theta+\sin\theta)\big|_0^{\sqrt{2}} = \frac{2\sqrt{2}}{3}(\cos\theta+\sin\theta)$.

Outer: $\frac{2\sqrt{2}}{3}\int_0^{\pi/4}(\cos\theta+\sin\theta)\,d\theta = \frac{2\sqrt{2}}{3}\left[\sin\theta - \cos\theta\right]_0^{\pi/4} = \frac{2\sqrt{2}}{3}(0-(-1)) = \frac{2\sqrt{2}}{3}$.