Math 53 Homework 7

Due Tuesday 10/18/11 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Chapter 14 – Complements

These two problems complete the material from Chapter 14, but are not directly related to the material on Midterm 1.

- **Work:** Problems 1 and 2 below.

Wednesday 10/12 – Double integrals

- **Read:** sections 15.1, 15.2, 15.3.
- **Work:** 15.2: (5), 12, 17, 21, (24), (25), (31).
  15.3: (1), 3, 6, 8, (13), 15, (21), 23, (33), 39, (41), (45), 47, (55), 58.

Friday 10/14 – Double integrals in polar coordinates

- **Read:** section 15.4.
- **Work:** 15.4: 6, 7, (11), (13), 17, (22), 25, (27), (29), 31, (35).

Problem 1. – Least-squares interpolation.

In experimental sciences, statistics, and many other fields, one often wishes to establish a linear relationship between two quantities (say x and y). Repeated experiments (or sampling of x and y for various individuals among the general population) give a set of experimental data (x_1, y_1), ..., (x_n, y_n) (each pertaining to a different experiment or to a different individual). One then attempts to find the straight line $y = mx + b$ that best fits the given experimental data. Least-squares interpolation is the most common method for doing so. (Note: the goal is to find m and b, which describe the relation between x and y – we are *not* trying to solve for x and y!). We will first work out the general formula (following a problem in the book), then apply it in an example.

a) Do section 14.7 exercise # 55. (You don't need to prove that the critical point is a minimum.)

Hint: In this problem, the total square deviation $\sum_{i=1}^{n} d_i^2$ is a function of the two variables m and b; all the x_i and y_i are constants (as part of the given experimental data). So this is a min-max problem in two variables, just like those we have seen in class. Looking for a critical point should give you the two equations. Proving that the critical point is indeed a minimum can be done (using the second derivative test or other methods) but takes quite a bit of effort; doing it is strictly optional.

b) The total number of SMS text messages (in billions) sent in the US during the month of December of each year from 2005 to 2010 is given in the table below (source: CTIA reports):

Years (x_i)	'05	'06	'07	'08	'09	'10
Messages (billions) (y_i)	9.8	18.7	48.1	110.4	152.7	187.7

To make the calculations easier, we take x to be the year minus 2000, so x_i ranges from 5 to 10 for the given data points. In this question, we take y_i to be the number of text messages sent in billions. So $(x_1, y_1) = (5, 9.8), \ldots, (x_6, y_6) = (10, 187.7)$.

Use the result of part (a) to write down the equations satisfied by the slope m and the intercept b of the best-fit line for this data. Next, solve these equations and give the values of m and b. (Use a calculator or a computer!)

c) Compare the predicted values $y = mx + b$ with the actual data for $x = 5$, $x = 7$ and $x = 9$. How good is the linear fit?

Optional: produce a plot that shows the given data points and the best fit curve.

d) According to the best fit, how many text messages will be sent in December 2011? in December 2020?

Problem 2. – Non-independent variables.

The goal of this problem is to illustrate a subtlety in the definition of partial derivatives when variables are not independent. This is an important issue in thermodynamics and some other fields; here we consider just a simple mathematical example.

Let $w = x^2 + y^2 + z^2$, where the variables x, y, z are related to each other by the equation $y^2 + xz = 2$. We can give three different meanings to the quantity $\partial w/\partial x$.

(i) We can treat the variables x, y, z as independent, and write $w = f(x, y, z) = x^2 + y^2 + z^2$. Then we consider $\partial f/\partial x$.

(ii) We can treat x and y as independent variables, with z implicitly defined as a function of x and y by the relation $y^2 + xz = 2$. Then w is given by some function $g(x, y)$, and we consider $\dfrac{\partial g}{\partial x}$. This quantity is sometimes denoted by $\left(\dfrac{\partial w}{\partial x}\right)_y$.

(iii) We can treat x and z as independent variables, with y implicitly defined as a function of x and z by the relation $y^2 + xz = 2$. Then w is given by some function $h(x, z)$, and we consider $\dfrac{\partial h}{\partial x}$. This quantity is sometimes denoted by $\left(\dfrac{\partial w}{\partial x}\right)_z$.

a) Determine the functions f, g, h, and calculate $\partial f/\partial x$, $\partial g/\partial x$, and $\partial h/\partial x$. Also say in each case which quantities are being held constant and which ones are not.

(Optional: compare the values of these partial derivatives at $(x, y, z) = (1, 1, 1)$ to convince yourself that they are really different.)

b) Now we try a more systematic approach, which would work even if we were unable to find expressions for the functions $g(x, y)$ and $h(x, z)$ by solving the constraint equation.

First, express the differential dw in terms of dx, dy and dz, and also differentiate the constraint equation to find a relation between dx, dy and dz. Then, use this relation to eliminate dz and express dw in terms of dx and dy; use this to find $\left(\dfrac{\partial w}{\partial x}\right)_y$. Similarly, eliminate dy to find $\left(\dfrac{\partial w}{\partial x}\right)_z$.

(Your answers might be different from those in part (a), but are they consistent with them?)