# Math 53 Homework 6 – Solutions

**14.8 # 3:** $f(x,y) = x^2 + y^2$, $g(x,y) = xy = 1 \Rightarrow \nabla f = \langle 2x, 2y\rangle$ and $\nabla g = \langle y, x\rangle$. Hence $\nabla f = \lambda \nabla g$ becomes: $2x = \lambda y$, $2y = \lambda x$. Rewriting the second equation as $y = \frac{\lambda}{2}x$, the first equation becomes $2x = \frac{1}{2}\lambda^2 x$, which implies $\lambda^2 = 4$ (note that $x \neq 0$ since $xy = 1$). So $\lambda = \pm 2$.

For $\lambda = 2$ we get $x = y$, and the constraint $xy = 1$ becomes $x^2 = 1$, so $x = \pm 1$, giving us the two points $(-1,-1)$ and $(1,1)$. For $\lambda = -2$ we get $x = -y$, and the constraint $xy = 1$ becomes $x^2 = -1$, no solution.

Geometrically, $(-1,-1)$ and $(1,1)$ are both minima of $f(x,y) = x^2+y^2$ (square of the distance from the origin) on the hyperbola $xy = 1$ (and $f(1,1) = f(-1,-1) = 2$). The maximum value is not attained, as $f$ tends to infinity when either $x \to 0$, $|y| \to \infty$ or $|x| \to \infty$, $y \to 0$.

**14.8 # 9:** $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6 \Rightarrow \nabla f = \langle yz, xz, zy$, $\nabla g = \langle 2x, 4y, 6z\rangle$. Thus $\nabla f = \lambda \nabla g$ becomes:

$$yz = 2\lambda x, \quad xz = 4\lambda y, \quad xy = 6\lambda z. \tag{1}$$

Multiplying the first equation by $x$, the second by $y$, the third by $z$, we get $xyz = 2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2$. There are two cases.

If $\lambda = 0$, then (1) becomes $yz = 0$, $xz = 0$, $xy = 0$, so two of $x, y, z$ must be zero. This gives the six points $(\pm\sqrt{6}, 0, 0)$, $(0, \pm\sqrt{3}, 0)$, $(0, 0, \pm\sqrt{2})$, at which $f = 0$.

Otherwise, $2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2$ implies $x^2 = 2y^2 = 3z^2$, and the constraint $x^2 + 2y^2 + 3z^2 = 6$ becomes $3x^2 = 6$, so $x^2 = 2$, $y^2 = 1$, $z^2 = \frac{2}{3}$. This gives the 8 points $(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}})$, at which $f = \pm\frac{2}{\sqrt{3}}$. Hence the maximum value of $f$ on the ellipsoid is $2/\sqrt{3}$, occurring at the 4 of these 8 points where all coordinates are positive or two are negative; and the minimum value is $-2/\sqrt{3}$, occurring at the 4 of these 8 points where one or three coordinates are negative.

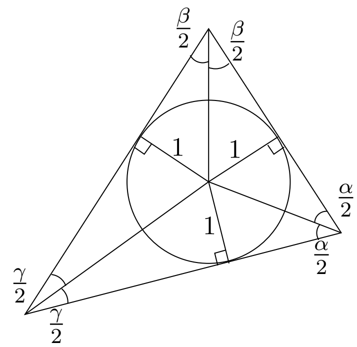
**14.8 # 29:** we want to minimize $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$ (the squared distance) subject to the constraint $g(x,y,z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda \nabla g$ becomes

$$2(x-4) = 2\lambda x, \quad 2(y-2) = 2\lambda y, \quad 2z = -2\lambda z.$$

Considering the last equation, either $z = 0$ (in which case the equation of the cone gives $x = y = 0$, which do not satisfy the other multiplier equations), or $\lambda = -1$. Plugging $\lambda = -1$ into the first two equations, we get $2(x-4) = -2x$ and $2(y-2) = -2y$. So $x = 2$, $y = 1$, and using the equation of the cone, $z = \pm\sqrt{5}$.

**14.8 # 35:** Let $f(x,y,z) = xyz$ (volume of the box), $g(x,y,z) = x + 2y + 3z = 6$ (the constraint). Then $\nabla f = \langle yz, xz, xy\rangle = \lambda \nabla g = \langle \lambda, 2\lambda, 3\lambda\rangle$. Then $\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy$ implies that either $\lambda = 0$ and two of $x, y, z$ are zero (but then $f = 0$, not a maximum); or $y = \frac{1}{2}x$ and $z = \frac{1}{3}x$. Plugging into the constraint equation, we get $3x = 6$, so $x = 2$, $y = 1$, and $z = \frac{2}{3}$. The largest volume is $f(2, 1, \frac{2}{3}) = \frac{4}{3}$.

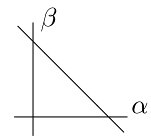
**Problem 1.** a) The triangle splits into 6 right triangles, two with area $\frac{1}{2}\cot\frac{\alpha}{2}$, two with area $\frac{1}{2}\cot\frac{\beta}{2}$, and the last two with area $\frac{1}{2}\cot\frac{\gamma}{2}$. Hence




$$A = \cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2}.$$

$\alpha+\beta+\gamma = \pi \Rightarrow \cot\frac{\gamma}{2} = \cot(\frac{\pi}{2} - \frac{\alpha+\beta}{2}) = \tan(\frac{\alpha+\beta}{2})$.
So $A = \cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \tan\frac{\alpha+\beta}{2}$.

b) The set of possible values for the angles is given by $\alpha > 0$, $\beta > 0$, $\alpha+\beta < \pi$. This is a triangular region in the $(\alpha,\beta)$ coordinates, with boundaries $\alpha = 0$, $\beta = 0$, and $\alpha+\beta = \pi$.




$$\frac{\partial A}{\partial \alpha} = -\frac{1}{2}\csc^2\frac{\alpha}{2} + \frac{1}{2}\sec^2\frac{\alpha+\beta}{2} \quad \text{and} \quad \frac{\partial A}{\partial \beta} = -\frac{1}{2}\csc^2\frac{\beta}{2} + \frac{1}{2}\sec^2\frac{\alpha+\beta}{2}.$$

So the critical points are solutions of $-\csc^2\frac{\alpha}{2} = \csc^2\frac{\beta}{2} = \sec^2\frac{\alpha+\beta}{2}$, or equivalently $\sin^2\frac{\alpha}{2} = \sin^2\frac{\beta}{2} = \cos^2\frac{\alpha+\beta}{2}$. Since these angles are between 0 and $\pi/2$, this implies $\alpha/2 = \beta/2 = \pi/2 - (\alpha+\beta)/2$. Hence the only critical point is $\alpha = \beta = \pi/3$.

c) At the critical point $\alpha = \beta = \frac{\pi}{3}$ we have $A = 3\sqrt{3}$. The boundary of the (triangular) domain where $A$ is defined consists of three parts: near the boundary $\alpha = 0$, the term $\cot\frac{\alpha}{2}$ increases to infinity. Similarly, $\cot\frac{\beta}{2}$ increases to infinity when $\beta$ approaches 0. Finally, near $\alpha+\beta = \pi$ the last term $\tan\frac{\alpha+\beta}{2}$ increases to infinity. So, no matter which boundary of the domain we consider, the value of $A$ increases to infinity as we approach it.
By comparing the values of $A$ at the critical point and near the boundary, the minimum of $A$ is $3\sqrt{3}$, reached for $\alpha = \beta = \gamma = \frac{\pi}{3}$, corresponding to an equilateral triangle. On the other hand, the value of $A$ can become arbitrarily large when one of $\alpha$, $\beta$ or $\gamma$ approaches zero, corresponding to the degenerate case of a very long and thin triangle, with two sides almost parallel to each other and intersecting at a vertex that lies very far from the incenter.

**Problem 2.** a) At $(\frac{3}{2},\frac{1}{2})$, changing $x$ while keeping $y$ fixed results in lower values of $f$ if $x$ increases, and higher values if $x$ decreases, so $f_x < 0$. Similarly, $f_y > 0$.

At the point $(1,1)$, $f_x < 0$ (same argument); and $f_y = 0$ (the line parallel to the $y$-axis through $(1,1)$ is tangent to the level curve $f = 0$, and the value of $f(1,y)$ passes through a maximum for $y = 1$, so at this point we have $f_y = 0$).

b) The directional derivative $D_{\hat{u}}f = \nabla f \cdot \hat{u}$ is zero when $\hat{u}$ is perpendicular to $\nabla f$, i.e. tangent to the level curve through $(\frac{3}{2},\frac{1}{2})$ (whose shape we can estimate from the neighboring ones). Hence, the two directions in which $D_{\hat{u}}f = 0$ are the two unit vectors along the tangent line to the level curve at $(\frac{3}{2},\frac{1}{2})$. (One is about 30° counterclockwise from $\hat{\imath}$, the other is in the opposite direction i.e. about 150° clockwise from $\hat{\imath}$).

c) The two critical points where $f_x = f_y = 0$ are near $P_1 = (0.3, 0.7)$ and $P_2 = (1.5, 1.25)$. The level curve through $P_1$ only consists of $P_1$ itself (for slightly lower values of $f$, the level curves are small ovals, shrinking to the point $P_1$ as one approaches the local maximum). The level curve through $P_2$ has two branches that intersect each other at $P_2$. Starting from $P_1$, the value of $f$ decreases in every direction. $P_1$ is a local maximum. Starting from $P_2$, the value of $f$ decreases if we move North or South, but increases if we move East or West. $P_2$ is a saddle point.

**Problem 3.** a) $f_x = 3x^2 - 6x + y + 1$, and $f_y = x - 2y + 1$, so $f_x(\frac{3}{2}, \frac{1}{2}) = -\frac{3}{4}$, $f_y(\frac{3}{2}, \frac{1}{2}) = \frac{3}{2}$; $f_x(1,1) = -1$, $f_y(1,1) = 0$.

b) From part (a), $\nabla f(\frac{3}{2}, \frac{1}{2}) = \langle -\frac{3}{4}, \frac{3}{2}\rangle$. The directions in which $D_{\hat{u}}f = \nabla f \cdot \hat{u} = 0$ are perpendicular to $\langle -\frac{3}{4}, \frac{3}{2}\rangle$. Rotating this vector by 90° in either direction gives us the two vectors $\langle \frac{3}{2}, \frac{3}{4}\rangle$ and $\langle -\frac{3}{2}, -\frac{3}{4}\rangle$; or, scaling down to unit length, the corresponding unit vectors are $\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$ and $\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\rangle$.

c) $f_y = x - 2y + 1 = 0$ gives $x = 2y - 1$, and then $f_x = 3x^2 - 6x + y + 1 = 0$ can be rewritten as: $3(2y-1)^2 - 6(2y-1) + y + 1 = 0$, or $12y^2 - 23y + 10 = 0$. The two roots of this quadratic are $(23 \pm \sqrt{49})/24$, i.e. $\frac{2}{3}$ and $\frac{5}{4}$. Remembering that $x = 2y - 1$, this gives the two critical points $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{3}{2}, \frac{5}{4})$.

Next we calculate $f_{xx} = 6x - 6$, $f_{xy} = f_{yx} = 1$, and $f_{yy} = -2$.

At $(\frac{1}{3}, \frac{2}{3})$: $f_{xx} = -4$, $f_{xy} = 1$, $f_{yy} = -2$, so $f_{xx}f_{yy} - f_{xy}^2 = (-4)(-2) - 1^2 = 7 > 0$, and $f_{xx} < 0$: hence we have a local maximum.

At $(\frac{3}{2}, \frac{5}{4})$: $f_{xx} = 3$, $f_{xy} = 1$, $f_{yy} = -2$, so $f_{xx}f_{yy} - f_{xy}^2 = 3(-2) - 1^2 = -7 < 0$, hence we have a saddle point.

**Problem 4.**

a) $\nabla g = \langle g_x, g_y, g_z\rangle = \langle 2x, 2y, -6\rangle = \langle 8, 4, -6\rangle$ at $(4,2,3)$. The direction of greatest decrease is that of $-\nabla g$, i.e. the unit vector $-\dfrac{\nabla g}{|\nabla g|} = \dfrac{\langle -8, -4, 6\rangle}{\sqrt{116}} = \dfrac{\langle -4, -2, 3\rangle}{\sqrt{29}}$.

b) Let $\Delta x = x - 4$, $\Delta y = y - 2$, $\Delta z = z - 3$; then the line in the direction of $\langle -4, -2, 3\rangle$ can be parametrized by $\Delta x = -4t$, $\Delta y = -2t$, $\Delta z = 3t$. (Dividing by $\sqrt{29}$ is unnecessary and makes calculations more complicated.) At $P_0 = (4,2,3)$, we have $g = 2$, and $\nabla g = \langle 8, 4, -6\rangle$ (from part (a)), so linear approximation gives

$$\begin{aligned} g(x,y,z) \approx g(P_0) + \nabla g(P_0) \cdot \langle \Delta x, \Delta y, \Delta z\rangle &= 2 + 8\Delta x + 4\Delta y - 6\Delta z \\ &= 2 + 8(-4t) + 4(-2t) - 6(3t) = 2 - 58t. \end{aligned}$$

Therefore, $g = 0$ when $2 - 58t \approx 0$, or $t \approx 1/29$. At $t = 1/29$, $(x,y,z) = (4 - 4t, 2 - 2t, 3 + 3t) = (4 - \frac{4}{29}, 2 - \frac{2}{29}, 3 + \frac{3}{29})$. Evaluating $g$ at this point, we find $\approx 0.024$, fairly close to 0.

**Problem 5.** a) Minimizing $f(x,y,z) = (x-4)^2 + (y-2)^2 + (z-3)^2$ subject to the constraint $g(x,y,z) = x^2 + y^2 - 6z = 0$ gives the Lagrange multiplier equations $\nabla f = \lambda \nabla g$, or

$$\begin{aligned} 2(x-4) &= \lambda(2x) \\ 2(y-2) &= \lambda(2y) \\ 2(z-3) &= \lambda(-6) \end{aligned}$$

The system now has four variables, so we also need to remember a fourth equation, the constraint equation, $x^2 + y^2 - 6z = 0$.

b)
$$\begin{aligned} x-4=\lambda x &\iff (1-\lambda)x=4 \iff x=4/(1-\lambda) \\ y-2=\lambda y &\iff (1-\lambda)y=2 \iff y=2/(1-\lambda) \\ z-3=-3\lambda &\iff z=3(1-\lambda) \end{aligned}$$

Thus $x=\dfrac{12}{z}$, $y=\dfrac{6}{z}$, and the constraint equation becomes $\left(\dfrac{12}{z}\right)^2+\left(\dfrac{6}{z}\right)^2-6z=0$. So $180/z^2=6z$, or $z^3=30$: so $z=\sqrt[3]{30}\approx 3.10723$. Using $x=12/z$ and $y=6/z$ one finds

$$(x,y,z)=\left(\tfrac{12}{\sqrt[3]{30}},\tfrac{6}{\sqrt[3]{30}},\sqrt[3]{30}\right)\approx(3.86196,1.93098,3.10723).$$

The answer found in Problem 3 was $(4-\frac{4}{29},2-\frac{2}{29},3+\frac{3}{29})\approx(3.86207,1.93103,3.10345)$ which is within $1/100$ of the exact solution (the largest difference is $\Delta z\approx 0.00378$).

**Answers to Problem 6.**

a) Denoting by $V$ the fixed volume and by $A$ the area of the base (also fixed), the height $h$ of the pyramid is determined by $V=\frac{1}{3}Ah$. But then the coordinates of $P$ must satisfy $z=h$.

Denoting by $P_1,P_2,P_3$ the three vertices of the base triangle, the quantity to minimize is $\frac{1}{2}|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P}|+\frac{1}{2}|\overrightarrow{P_2P_3}\times\overrightarrow{P_2P}|+\frac{1}{2}|\overrightarrow{P_3P_1}\times\overrightarrow{P_3P}|$, which once expressed in coordinates becomes a very complicated expression. A computer algebra system can find the minimum, but the answer will be very hard to interpret.

b) The area $A$ of the base triangle $P_1P_2P_3$ is the sum of the areas of $P_1P_2Q$, $P_2P_3Q$ and $P_3P_1Q$. These smaller triangles have base $a_i$ and height $u_i$, so the constraint is $g(u_1,u_2,u_3)=\frac{1}{2}a_1u_1+\frac{1}{2}a_2u_2+\frac{1}{2}a_3u_3=A$.

Denoting by $h$ the height of the pyramid (determined by $A$ and the volume, see (a)), each side face is a triangle with base $a_i$ and height $\sqrt{u_i^2+h^2}$. So the quantity to minimize is $f(u_1,u_2,u_3)=\frac{1}{2}a_1\sqrt{u_1^2+h^2}+\frac{1}{2}a_2\sqrt{u_2^2+h^2}+\frac{1}{3}a_3\sqrt{u_3^2+h^2}$.

c) $\partial f/\partial u_i=\frac{1}{2}a_iu_i(u_i^2+h^2)^{-1/2}$, while $\partial g/\partial u_i=\frac{1}{2}a_i$. So $\nabla f=\lambda\nabla g$ yields:
$$\frac{u_1}{\sqrt{u_1^2+h^2}}=\frac{u_2}{\sqrt{u_2^2+h^2}}=\frac{u_3}{\sqrt{u_3^2+h^2}}=\lambda.$$
Since the quantity $\dfrac{u}{\sqrt{u^2+h^2}}$ is a monotonically increasing function of $u$, this implies that $u_1=u_2=u_3$. Hence $Q$ is at the same distance from all three sides of the triangle, i.e. it lies at the incenter. (In conclusion: the apex of the pyramid should lie directly above the incenter).