Math 53 Homework 5 – Solutions

14.5 # 1: $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2x+y)\cos t + (2y+x)e^t = (2\sin t + e^t)\cos t + (2e^t + \sin t)e^t$.

14.5 # 7: $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = 2xy^3\,\cos t + 3x^2y^2\,\sin t$.

$\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = -2xy^3\,s\sin t + 3x^2y^2\,s\cos t$.

14.5 # 13: $z = f(g(t), h(t))$, so $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dg}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dh}{dt} = f_x(x,y)\,g'(t) + f_y(x,y)\,h'(t)$. At $t=3$, $(x,y) = (2,7)$; using the given values, $dz/dt = (6)(5) + (-8)(-4) = 62$.

14.5 # 19: $\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial w}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial x}$;

$\dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial w}{\partial s}\dfrac{\partial s}{\partial y} + \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial y}$.

14.5 # 43: Call the sides a and b, and the angle between them θ; so the area $A = \frac{1}{2}ab\sin\theta$ is assumed to be constant. So $\dfrac{dA}{dt} = \dfrac{\partial A}{\partial a}\dfrac{da}{dt} + \dfrac{\partial A}{\partial b}\dfrac{db}{dt} + \dfrac{\partial A}{\partial \theta}\dfrac{d\theta}{dt} = 0$, i.e., $\frac{1}{2}b\sin\theta\,\dfrac{da}{dt} + \frac{1}{2}a\sin\theta\,\dfrac{db}{dt} + \frac{1}{2}ab\cos\theta\,\dfrac{d\theta}{dt} = 0$. Solving for $d\theta/dt$, we get: $\dfrac{d\theta}{dt} = -\dfrac{b\sin\theta\, da/dt + a\sin\theta\, db/dt}{ab\cos\theta}$. Using the given values for $a, b, \theta, da/dt, db/dt$, we get $\dfrac{d\theta}{dt} = -\dfrac{30\cdot\frac{1}{2}\cdot 3 + 20\cdot\frac{1}{2}\cdot(-2)}{20\cdot 30\cdot\frac{\sqrt{3}}{2}} = -\frac{5}{60\sqrt{3}} \approx -0.048$ rad/s.

14.5 # 45: (a) chain rule: $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}\,r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$, and $\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. So $\left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$.

14.5 # 51: $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = 2s\dfrac{\partial z}{\partial x} + 2r\dfrac{\partial z}{\partial y}$. So

$$\frac{\partial^2 z}{\partial r\partial s} = \frac{\partial}{\partial r}\left(2s\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial y}\right) = 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) + 2\frac{\partial z}{\partial y}$$

$$= 2s\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\right) + 2r\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r}\right) + 2\frac{\partial z}{\partial y}$$

$$= 4rs\frac{\partial^2 z}{\partial x^2} + 4s^2\frac{\partial^2 z}{\partial y\partial x} + 4r^2\frac{\partial^2 z}{\partial x\partial y} + 4rs\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial y}$$

$$= 4rs\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + 4(r^2+s^2)\frac{\partial^2 z}{\partial x\partial y} + 2\frac{\partial z}{\partial y}.$$

Explanation: to compute $\partial^2 z/\partial r\partial s$, we differentiate the expression for $\partial z/\partial s$ with respect to r (keeping s constant). The first step involves the product rule. The second one is more subtle. To calculate $\frac{\partial}{\partial r}(\frac{\partial z}{\partial x})$, we use the chain rule once more. If this feels confusing, just set $g = \partial z/\partial x$ (think of this as some new function of x and y), and note that we are trying to calculate $\frac{\partial g}{\partial r}$. The chain rule gives $\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r}$. Remembering that $g = \partial z/\partial x$, the first partials of g with respect to x and y are actually second partial derivatives of z. The chain rule is used similarly to differentiate $\partial z/\partial y$ with respect to r.

14.6 # 1: We can approximate the directional derivative at K by the average rate of change of pressure between the points where the line through K and S (red on the figure) intersects the contour lines closest to K. In this case we measure that, going about 1/6 of the way towards S, ($\Delta s \approx 50$ km), the pressure drops from 1000 to 996 mb ($\Delta P \approx -4$ mb). So $D_{\hat{u}}P \approx \frac{\Delta P}{\Delta s} \approx \frac{-4}{50} = -0.08$ (millibars per km).

14.6 # 9: (a) $f(x,y,z) = xe^{2yz} \Rightarrow \nabla f = \langle f_x, f_y, f_z\rangle = \langle e^{2yz}, 2xze^{2yz}, 2xye^{2xz}\rangle$.
(b) at $(x,y,z) = (3,0,2)$, $\nabla f = \langle 1, 12, 0\rangle$.
(c) $\hat{u} = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\rangle$ is a unit vector, so $D_{\hat{u}}f = \nabla f \cdot \hat{u} = \langle 1,12,0\rangle \cdot \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\rangle = -\frac{22}{3}$.

14.6 # 25: $\nabla f(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$.
$\nabla f(3,6,-2) = \langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\rangle$ gives the direction of maximum rate of change, and the maximum rate is $D_{\mathrm{dir}(\nabla f)}f = |\nabla f| = 1$.
(Note: since $f(x,y,z)$ is the distance from the origin to (x,y,z), the answer makes sense geometrically: distance from the origin increases fastest when moving radially outwards, and the rate of increase is 1.)

14.6 # 27: (a) Given a unit direction vector $\hat{u}$, recall that $D_{\hat{u}}f = \nabla f \cdot \hat{u} = |\nabla f| \cos\theta$ (where θ is the angle between ∇f and $\hat{u}$. Since the minimum value of $\cos\theta$ is -1, occurring for $\theta = \pi$, the minimum value of $D_{\hat{u}}f$ is $-|\nabla f|$ and occurs when $\hat{u}$ is in the opposite direction of ∇f.
(b) $\nabla f = \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2\rangle$, so at the point $(2,-3)$ f decreases fastest in the direction of $-\nabla f(2,-3) = -\langle 12, -92\rangle = \langle -12, 92\rangle$ (or the corresponding unit vector).

14.6 # 38: $\nabla f(4,6)$ is perpendicular to the level curve of f that passes through $(4,6)$; so we sketch a portion of level curve through $(4,6)$ (using the nearby level curves as guidelines), and draw a line perpendicular to it. The direction of the gradient vector is parallel to this line, and pointing towards increasing values of f. (towards the lower-right, making about a 65° angle with the horizontal direction).

Next we estimate the magnitude $|\nabla f|$, which equals the directional derivative of f at (4,6) in the direction of ∇f. We estimate this by finding the average rate of change along the direction perpendicular to the level curve. The points where the line previously drawn intersects the contour lines $f = -2$ and $f = -3$ are ≈ 0.5 units apart, so $\Delta f = 1$ and $\Delta s \approx 0.5$, giving $|\nabla f| \approx \frac{\Delta f}{\Delta s} \approx \frac{1}{0.5} = 2$. Hence we sketch the gradient vector with length 2. (Diagram omitted).

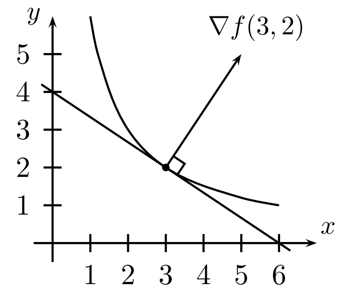
(Note: we could also have tried to estimate f_x and f_y separately and use those to sketch ∇f; this is much less accurate, especially concerning the direction of ∇f.)

14.6 # 41: Let $f(x, y, z) = x^2 - 2y^2 + z^2 + yz$: then we are considering the level surface $f = 2$. Moreover, $\nabla f = \langle 2x, -4y + z, 2z + y\rangle$, so $\nabla f(2, 1, -1) = \langle 4, -5, -1\rangle$.

(a) $\nabla f(2, 1, -1) = \langle 4, -5, -1\rangle$ is a normal vector for the tangent plane at $(2, 1, -1)$, so an equation of the tangent plane is $4(x-2)-5(y-1)-1(z+1) = 0$ or $4x-5y-z = 4$.

(b) The normal line has direction $\langle 4, -5, -1\rangle$, so parametric equations are $x = 2+4t$, $y = 1 - 5t$, $z = -1 - t$.

14.6 # 47: $\nabla f = \langle y, x\rangle$, so $\nabla f(3, 2) = \langle 2, 3\rangle$. So the tangent line has equation $\langle 2, 3\rangle \cdot \langle x - 3, y - 2\rangle = 0 \Rightarrow 2(x - 3) + 3(y - 2) = 0$, which simplifies to $2x + 3y = 12$.



14.6 # 54: first note that the point $(1, 1, 2)$ is on both surfaces. Let $f(x, y, z) = 3x^2 + 2y^2 + z^2$, so the ellipsoid is $f = 9$: then $\nabla f = \langle 6x, 4y, 2z\rangle$, so the tangent plane to the ellipsoid has normal vector $\nabla f(1, 1, 2) = \langle 6, 4, 4\rangle$, and an equation of the tangent plane is $6x+4y+4z = 18$ or $3x+2y+2z = 9$. The sphere is the level surface $g = 0$ where $g(x, y, z) = x^2+y^2+z^2-8x-6y-8z+24$, and $\nabla g = \langle 2x-8, 2y-6, 2z-8\rangle$. So the tangent plane at (1,1,2) has normal vector $\nabla g(1, 1, 2) = \langle -6, -4, -4\rangle$, giving the equation $-6x - 4y - 4z = -18$ or $3x + 2y + 2z = 9$. The tangent planes are the same, so the surfaces are tangent to each other at (1,1,2).

(Note: it would have been enough to show that the normal vectors are parallel to each other, without determining the equations of the tangent planes.)

14.7 # 3: From the contour plot, there appears to be a local minimum near $(1, 1)$ (enclosed by oval-shaped level curves indicating that as we move away from the point in any direction the values of f are increasing). Moreover, the shape of the level curves near the origin is characteristic of a saddle point at $(0, 0)$.

To verify these guesses, we have $f(x, y) = 4+x^3+y^3-3xy \Rightarrow f_x = 3x^2-3y$ and $f_y = 3y^2 - 3x$. We have critical points when $f_x = f_y = 0$. The first equation $3x^2 - 3y = 0$ gives $y = x^2$, and substituting into the second equation gives $3(x^2)^2 - 3x = 0$, hence $3x(x^3 - 1) = 0$, hence $x = 0$ or $x = 1$. So we have two critical points $(0, 0)$ and $(1, 1)$.

The second partial derivatives are $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$, so $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9$. Then $D(0, 0) = 0 - 9 < 0$ so f has a saddle point at (0,0); and $D(1, 1) = 36 - 9 > 0$, with $f_{xx}(1, 1) = 6 > 0$, so f has a local minimum at $(1, 1)$.

14.7 # 7: $f(x, y) = x^4 + y^4 - 4xy + 2 \Rightarrow f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$, $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$. Then $f_x = 0 \iff y = x^3$, $f_y = 0 \iff x = y^3$; substituting, we get $x = x^9$, or $x(x^8 - 1) = 0$, which gives the three solutions $x = 0$ or $x = \pm 1$. Thus the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

$D(0, 0) = 0 \cdot 0 - (-4)^2 = -16 < 0$, so the origin is a saddle point. $D(1, 1) = (12)(12) - (-4)^2 > 0$ and $f_{xx}(1, 1) = 12 > 0$, so $(1, 1)$ is a local minimum (with value $f(1,1)=0$). Similarly, $D(-1,-1)=(12)(12)-(-4)^2>0$ and $f_{xx}(-1,-1)=12>0$, so $(-1,-1)$ is also a local minimum, with $f(-1,-1)=0$.

14.7 # 33: By 14.7 # 7, the only critical point of f inside D is $(1,1)$, a local minimum with $f(1,1)=0$. Next we consider the boundaries.

For $y=0$, $0\le x\le 3$ (bottom edge): $f(x,0)=x^4+2$, attaining its minimum at $x=0$ ($f(0,0)=2$) and its maximum at $x=3$ ($f(3,0)=83$).

For $x=3$, $0\le y\le 2$ (right edge): $f(3,y)=y^4-12y+83$, a polynomial which attains its minimum for $\frac{d}{dy}(y^4-12y+83)=4y^3-12=0$ or $y=\sqrt[3]{3}$, with $f(3,\sqrt[3]{3})=-9\sqrt[3]{3}+83\approx 70$, and its maximum at $y=0$, $f(3,0)=83$.

For $y=2$, $0\le x\le 3$ (top edge): $f(x,2)=x^4-8x+18$, attaining its minimum for $\frac{d}{dx}(x^4-8x+18)=4x^3-8x=0$ or $x=\sqrt[3]{2}$, with $f(\sqrt[3]{2},2)=-6\sqrt[3]{2}+18\approx 10.4$, and its maximum at $x=3$, $f(3,2)=75$ ($>f(0,2)=18$).

For $x=0$, $0\le y\le 2$ (left edge): $f(0,y)=y^4+2$, attaining its minimum at $x=0$ ($f(0,0)=2$) and its maximum at $y=2$ ($f(0,2)=18$).

Comparing, the absolute minimum of f on D is $f(1,1)=0$, and the absolute maximum is $f(3,0)=83$.

14.7 # 34: $f_x=y^2$ and $f_y=2xy$, and the critical points correspond to $y=0$; so there are none in the interior of D (the portion of the disk of radius $\sqrt{3}$ lying in the first quadrant). So the maximum and minimum occur at the boundaries. On the x and y axes, $f(x,0)=f(0,y)=0$. On the circular edge, $y=\sqrt{3-x^2}$ so we consider $g(x)=f(x,\sqrt{3-x^2})=3x-x^3$ for $0\le x\le\sqrt{3}$. Then $g'(x)=3-3x^2=0$ $\Leftrightarrow x=1$. We get that the maximum value on this edge is $g(1)=f(1,\sqrt{2})=2$, and the minimum value is 0, occurring both at $x=0$ and $x=\sqrt{3}$. Thus the absolute maximum of f on D is $f(1,\sqrt{2})=2$, and the absolute minimum is 0 which occurs at all points along the x and y axes.

14.7 # 41: We want to minimize the distance from $(4,2,0)$ to (x,y,z), $d=\sqrt{(x-4)^2+(y-2)^2+z^2}$, where $z^2=x^2+y^2$. Instead, it is easier to minimize $d^2=f(x,y)=(x-4)^2+(y-2)^2+(x^2+y^2)$. Since $f_x=2(x-4)+2x=4x-8$ and $f_y=2(y-2)+2y=4y-4$, the only critical point is $(x,y)=(2,1)$. This point must correspond to the minimum distance ($f(x,y)\to\infty$ when x and/or y become large). For $x=2$ and $y=1$, the equation of the cone gives $z^2=5$ or $z=\pm\sqrt{5}$. Hence the points on the cone closest to $(4,2,0)$ are $(2,1,\pm\sqrt{5})$.

14.7 # 47: Let (x,y,z) be the corner opposite the origin. Since $z=\frac{1}{3}(6-x-2y)$ and the volume is xyz, we want to maximize $f(x,y)=xyz=\frac{1}{3}xy(6-x-2y)$.

$f_x=\frac{1}{3}y(6-2x-2y)$, and $f_y=\frac{1}{3}x(6-x-4y)$. Setting $f_x=f_y=0$ gives $2x+2y=6$ and $x+4y=6$, hence the only critical point is $(x,y)=(2,1)$, which geometrically must yield a maximum. Thus the volume of the largest box is $V=f(2,1)=\frac{4}{3}$.