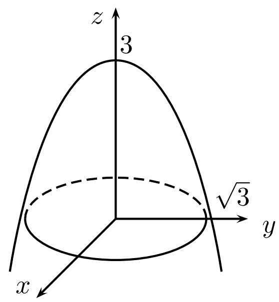
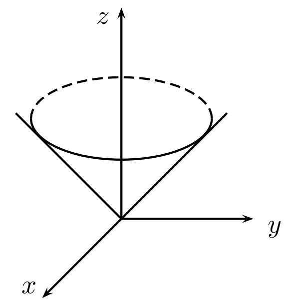
# Math 53 Homework 4 – Solutions

**14.1 # 26:** $z = 3 - x^2 - y^2$: a paraboloid with its highest point at $(0, 0, 3)$ and intersecting the $xy$-plane at the circle $x^2 + y^2 = 3$ of radius $\sqrt{3}$. (or: rotate the parabola $z = 3 - y^2$ in the $yz$-plane about the $z$-axis).

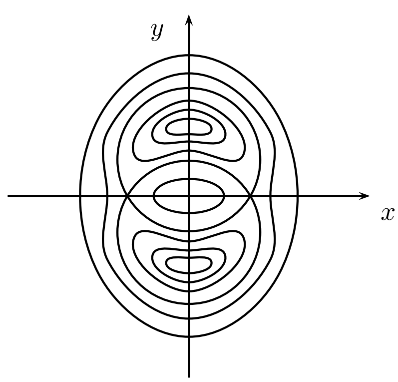



**14.1 # 29:** $z = \sqrt{x^2 + y^2}$: the top half of a right circular cone (the height $z$ is equal to the distance $\sqrt{x^2 + y^2}$ from the $z$-axis). (or: rotate the half-line $z = y$, $y \geq 0$ in the $yz$-plane about the $z$-axis).

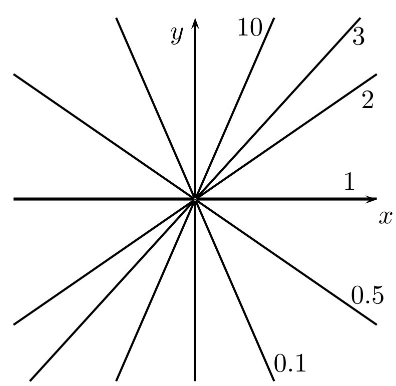



**14.1 # 32:** If we start at the origin and move along, say, the $x$-axis, the $z$-values of a cone with its vertex at the origin increase at a constant rate, so we expect the level curves to be equally spaced. A paraboloid centered on the $z$-axis, on the other hand, has $z$-values which change slowly near the origin and more quickly as we move further away; thus we expect the level curves to be spaced more widely apart near the origin and more closely together further away. (See pictures above). So contour map I must correspond to the paraboloid, and map II to the cone.

**14.1 # 34:** 2 maxima on the $y$-axis; 2 saddle points on the $x$-axis surrounding a local min at $(0, 0)$ (not directly visible but implied by the picture).




**14.1 # 42:** $e^{y/x} = c \Leftrightarrow y/x = \ln c$.
So the level curve $f(x, y) = c$ is a straight line with slope $\ln c$ through the origin.




**14.1 # 55:** graph C and contour map II: indeed, the function is the same if $x$ is interchanged with $y$, so the graph and contour map are symmetric about the plane $x = y$; moreover the function is constant over each hyperbola $xy$ =constant (since its value only depends on that of $xy$), and its values oscillate between $-1$ and 1.

**14.1 # 62:** the level sets $x^2 + 3y^2 + 5z^2 = c$ form a family of ellipsoids for $c > 0$ (and the origin for $c = 0$).

**14.2 # 9:** Let $f(x, y) = y^4/(x^4 + 3y^4)$. On the $x$-axis, $f(x, 0) = 0$ for $x \neq 0$, so $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the $x$-axis. On the other hand, approaching (0,0) along the $y$-axis, $f(0, y) = 1/3$, so $f(x, y) \to 1/3$ along the $y$-axis. Since $f$ has two different limits along two different lines, the limit does not exist.

**14.2 # 13:** $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$. We can see that the limit along any line through (0,0) is 0, as well as along various other paths approaching (0,0). So it is reasonable to expect that the limit exists and equals 0. One way to prove it is to notice that, since $|y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2}$, we have $0 \leq \left| \dfrac{xy}{\sqrt{x^2 + y^2}} \right| \leq |x|$. Since $|x| \to 0$ as $(x, y) \to (0, 0)$, this bound implies that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$.

(Alternative solution: in polar coordinates, $f(r \cos\theta, r \sin\theta) = \dfrac{(r \cos\theta)(r \sin\theta)}{r} = r \cos\theta \sin\theta$, so indeed $|f(x, y)| \leq r$ and $f(x, y) \to 0$ as $(x, y) \to (0, 0)$.)

**14.2 # 39:**

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r\to 0^+} \frac{(r \cos\theta)^3 + (r \sin\theta)^3}{r^2} = \lim_{r\to 0^+} (r \cos^3\theta + r \sin^3\theta) = 0.$$

**14.3 # 10:** Starting at $(2, 1)$, where $f(2, 1) = 10$, and moving in the positive $x$-direction, we reach the next contour line ($f = 12$) after approximately 0.6 units. This represents an average rate of change of $\frac{\Delta f}{\Delta x} \approx \frac{2}{0.6} \approx 3.3$. Or, moving in the negative $x$-direction, we reach the next contour line ($f = 8$) after about 0.8 units, representing an average rate of change of $\frac{\Delta f}{\Delta x} \approx \frac{-2}{-0.8} = 2.5$. Either of these (or even better, their average) would be a reasonable estimate of $f_x(2, 1)$.

Similarly for $f_y$: moving in the positive $y$-direction the value of $f$ decreases from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{\Delta f}{\Delta y} \approx \frac{-2}{0.9} \approx -2.2$. Or, moving in the negative $y$-direction, $f$ increases from 10 to 12 after about 1 unit, which corresponds to $\frac{\Delta f}{\Delta y} \approx \frac{2}{-1} = -2$. Either value is a reasonable estimate of $f_y(2, 1)$.

**14.3 # 21:** $f(x, y) = \dfrac{x - y}{x + y} \Rightarrow f_x = \dfrac{(1)(x + y) - (x - y)(1)}{(x + y)^2} = \dfrac{2y}{(x + y)^2}$ (by the quotient rule) and $f_y = \dfrac{(-1)(x + y) - (x - y)(1)}{(x + y)^2} = \dfrac{-2x}{(x + y)^2}$.

**14.3 # 40:** $f(x, y) = \tan^{-1}(\frac{y}{x})$: $f_x = \dfrac{-y}{x^2} \cdot \dfrac{1}{1 + (y/x)^2} = \dfrac{-y}{x^2 + y^2}$, so $f_x(2, 3) = \frac{-3}{13}$.

**14.3 # 45:** Differentiating $x^2 + y^2 + z^2 = 3xyz$ with respect to $x$ gives:

$$\frac{\partial}{\partial x}(x^2 + y^2 + z^2) = \frac{\partial}{\partial x}(3xyz), \text{ so } 2x + 0 + 2z\frac{\partial z}{\partial x} = 3yz + 3xy\frac{\partial z}{\partial x}.$$

(Note: $x$ and $y$ are independent variables, while $z$ is implicitly a function of $x, y$).

Therefore: $(2z - 3xy)\dfrac{\partial z}{\partial x} = 3yz - 2x$, so $\dfrac{\partial z}{\partial x} = \dfrac{3yz - 2x}{2z - 3xy}$.

Similarly, $\dfrac{\partial}{\partial y}(x^2 + y^2 + z^2) = \dfrac{\partial}{\partial y}(3xyz)$, so $2y + 2z\dfrac{\partial z}{\partial y} = 3xz + 3xy\dfrac{\partial z}{\partial y}$, which gives $\dfrac{\partial z}{\partial y} = \dfrac{3xz - 2y}{2z - 3xy}$.

**14.3 # 51:** $f(x,y) = x^3y^5 + 2x^4y$, so $f_x = 3x^2y^5 + 8x^3y$ and $f_y = 5x^3y^4 + 2x^4$. Differentiating $f_x$, we get $f_{xx} = 6xy^5 + 24x^2y$ and $f_{xy} = 15x^2y^4 + 8x^3$; differentiating $f_y$, we get $f_{yx} = 15x^2y^4 + 8x^3$ and $f_{yy} = 20x^3y^3$. (Note that $f_{xy} = f_{yx}$ as expected.)

**14.3 # 73:** $u(x,y,z) = (x^2+y^2+z^2)^{-1/2} \Rightarrow u_x = -\frac{1}{2}(2x)(x^2+y^2+z^2)^{-3/2} = -x(x^2+y^2+z^2)^{-3/2}$, and $u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\cdot(-\frac{3}{2})(2x)(x^2+y^2+z^2)^{-5/2} = \dfrac{-(x^2+y^2+z^2)+3x^2}{(x^2+y^2+z^2)^{5/2}} = \dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$.

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$; thus

$$u_{xx}+u_{yy}+u_{zz} = \frac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0.$$

(Note: if you've already taken Physics 7B, you might know that the electric potential solves the Laplace equation in regions of space that contain no electric charges. The calculation we just did confirms this for the electric potential of a charged particle at the origin, which is a constant multiple of $(x^2+y^2+z^2)^{-1/2}$.)

**14.3 # 75:** By the (single variable) chain rule, $\frac{\partial}{\partial x}(f(x+at)) = f'(x+at)$, while $\frac{\partial}{\partial t}(f(x+at)) = \frac{\partial(x+at)}{\partial t}f'(x+at) = af'(x+at)$. Similarly for the partial derivatives of $g(x-at)$, $\frac{\partial}{\partial x}(g(x-at)) = g'(x-at)$ and $\frac{\partial}{\partial t}(g(x-at)) = -ag'(x-at)$.

Given $u(x,t) = f(x+at)+g(x-at)$, we have $u_t = af'(x+at) - ag'(x-at)$, and, differentiating again, $u_{tt} = a^2f''(x+at) + a^2g''(x-at)$; similarly, $u_x = f'(x+at) + g'(x-at)$ and $u_{xx} = f''(x+at) + g''(x-at)$. So $u_{tt} = a^2u_{xx}$.

(Note: physically, $f(x+at)$ represents a wave whose shape is given by $f$ and which travels towards negative $x$ at speed $a$: indeed, for fixed $t$, the graph of $f(x+at)$ is obtained from that of $f(x)$ by shifting by $at$ to the left. Similarly, $g(x-at)$ represents a wave with profile $g$ and travelling towards positive $x$ at speed $a$. The given $u(x,t)$ is the superposition of a left-moving and a right-moving wave.)

**14.3 # 82:** Note: the equation $PV = mRT$ can be used to solve for any one of $P, V, T$ as a function of the two others. The notation $\partial P/\partial V$ means that we view $P$ as a function of $V$ and $T$, and consider the rate of change of $P$ with respect to $V$ (with $T$ fixed). Similarly for the other partials in the statement of the problem.

With this understood: $P = mRT/V$, so $\partial P/\partial V = -mRT/V^2$; $V = mRT/P$, so $\partial V/\partial T = mR/P$; and $T = PV/mR$, so $\partial V/\partial P = V/mR$. Multiplying, we get

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = \frac{-mRT}{V^2}\frac{mR}{P}\frac{V}{mR} = -\frac{mRT}{PV} = -1.$$

This apparently paradoxical equality is actually a special case of a more general identity (see 14.5 problem # 58). It illustrates the dangers of relying on notation (no, you can't simplify products of partial derivatives!). The reason why there is no paradox is that each individual quantity in the product corresponds to a *different* situation: $\partial P/\partial V$ assumes we vary $V$ (and hence $P$) keeping $T$ fixed, while for $\partial V/\partial T$ it is $P$ that is kept fixed, and for $\partial T/\partial P$ it is $V$ that remains constant.

**14.3 # 87:** If $f_x = x + 4y$ then $f_{xy} = 4$; however, if $f_y = 3x - y$ then $f_{yx} = 3$. Since $f_{xy}$ and $f_{yx}$ are continuous everywhere but $f_{xy}(x, y) \neq f_{yx}(x, y)$, the existence of such a function $f$ would contradict Clairaut's theorem.

**14.4 # 5:** Differentiating $z = f(x, y) = y\cos(x - y)$, we get $f_x = -y\sin(x - y)$ and $f_y = \cos(x - y) + y\sin(x - y)$; so $f_x(2, 2) = 0$ and $f_y(2, 2) = 1$, and the tangent plane is $z - 2 = 0(x - 2) + 1(y - 2)$, or $z = y$.

**14.4 # 17:** Let $f(x, y) = \dfrac{2x + 3}{4y + 1}$, so $f_x = 2/(4y+1)$ and $f_y = -4(2x+3)/(4y+1)^2$. Evaluating at $(0, 0)$: $f(0, 0) = 3$, $f_x(0, 0) = 2$ and $f_y(0, 0) = -12$. So the linear approximation is $f(0, 0) + f_x(0, 0) \cdot x + f_y(0, 0) \cdot y = 3 + 2x - 12y$.

**14.4 # 21:** $f_x = \frac{1}{2}(2x)(x^2 + y^2 + z^2)^{-1/2} = x(x^2 + y^2 + z^2)^{-1/2}$; $f_y = y(x^2 + y^2 + z^2)^{-1/2}$ and $f_z = z(x^2 + y^2 + z^2)^{-1/2}$. Evaluating at $(x, y, z) = (3, 2, 6)$, where $f = \sqrt{9 + 4 + 36} = 7$, we get $f_x(3, 2, 6) = 3/7$, $f_y(3, 2, 6) = 2/7$, and $f_z(3, 2, 6) = 6/7$. So the linear approximation is: $f(x, y, z) \approx 7 + \frac{3}{7}(x - 3) + \frac{2}{7}(y - 2) + \frac{6}{7}(z - 7)$.
Evaluating at $(3.02, 1.97, 5.99)$: $7+\frac{3}{7}(0.02)+\frac{2}{7}(-0.03)+\frac{6}{7}(-0.01) = 7-\frac{0.06}{7} \approx 6.9914$.

**14.4 # 28:** $T(u, v, w) = \dfrac{v}{1 + uvw}$, so $dT = \dfrac{\partial T}{\partial u}du + \dfrac{\partial T}{\partial v}dv + \dfrac{\partial T}{\partial w}dw =$
$\dfrac{-v \cdot vw}{(1 + uvw)^2}\,du + \dfrac{((1 + uvw) - v \cdot uw}{(1 + uvw)^2}\,dv + \dfrac{-v \cdot uv}{(1 + uvw)^2}\,dw = \dfrac{-v^2w\,du + dv - uv^2\,dw}{(1 + uvw)^2}$.

**14.4 # 33:** Let $x$ and $y$ be the two sides; then the area is $A(x, y) = xy$. Since $dA = \frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$, at $(x, y) = (30, 24)$ we have the linear approximation $\Delta A \approx y\Delta x + x\Delta y = 24\Delta x + 30\Delta y$. Given $|\Delta x| \leq 0.1$ and $|\Delta y| \leq 0.1$, the maximum error in the area is about $\Delta A \simeq 24(0.1) + 30(0.1) = 5.4$ cm$^2$.

**14.4 # 38:** Here $P = 8.31\,T/V$, so $dP = (8.31/V)\,dT - (8.31\,T/V^2)\,dV$, hence $\Delta P \approx 8.31\Big(\dfrac{\Delta T}{V} - \dfrac{T\Delta V}{V^2}\Big) = 8.31\Big(\dfrac{-5}{12} - \dfrac{310 \cdot 0.3}{12^2}\Big) \approx -8.83$.
The pressure will decrease by about 8.83 kPa.