Math 53 Homework 3 – Solutions

12.5 # 5: A line perpendicular to the given plane $x + 3y + z = 5$ must be parallel to a normal vector to the plane, for instance $\vec{n} = \langle 1, 3, 1\rangle$. So the vector equation is determined by the initial position $\vec{r}_0 = \langle 1, 0, 6\rangle$ and the direction vector $\vec{v} = \vec{n} = \langle 1, 3, 1\rangle$, giving $\vec{r} = \vec{r}_0 + t\vec{v} = \langle 1, 0, 6\rangle + t\langle 1, 3, 1\rangle = \langle 1+t, 3t, 6+t\rangle$. Hence, parametric equations are: $x = 1 + t$, $y = 3t$, $z = 6 + t$.

12.5 # 33: Call the given points P, Q, R: the vectors $\overrightarrow{PQ} = \langle 8-3, 2-(-1), 4-2\rangle = \langle 5, 3, 2\rangle$ and $\overrightarrow{PR} = \langle -1-3, -2-(-1), -3-2\rangle = \langle -4, -1, -5\rangle$ lie in the plane, so a normal vector is $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -13, 17, 7\rangle$, and an equation of the plane is $-13(x-3) + 17(y-(-1)) + 7(z-2) = 0$, or $-13x + 17y + 7z = -42$.

12.5 # 46: A direction vector for the line through $P_0 = (1, 0, 1)$ and $P_1 = (4, -2, 2)$ is $\vec{v} = \overrightarrow{P_0P_1} = \langle 3, -2, 1\rangle$, and parametric equations are $x = 1 + 3t$, $y = 0 - 2t$, $z = 1 + t$. Substituting these parametric equations into the equation of the plane gives $(1 + 3t) - 2t + (1 + t) = 6$, or $2 + 2t = 6$, hence $t = 2$. Plugging $t = 2$ into the parametric equations, we find that the intersection is at the point $(7, -4, 3)$.

12.5 # 56: (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z = 0$. (This will only work if the line of intersection crosses the xy-plane; otherwise try setting x or y equal to 0). Then the equations of the planes reduce to $3x - 2y = 1$ and $2x + y = 3$. Solving these two equations gives $x = 1$, $y = 1$. So a point on the line of intersection is $(1, 1, 0)$.

The direction of the line has to be contained in both planes, hence it should be perpendicular to both normal vectors $\vec{n}_1 = \langle 3, -2, 1\rangle$ and $\vec{n}_2 = \langle 2, 1, -3\rangle$. Hence

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} = \langle 5, 11, 7\rangle \text{ is parallel to the line.}$$

This yields the parametric equations $x = 1 + 5t$, $y = 1 + 11t$, $z = 7t$. (There are many other parametric equations of the same line.)

(b) We find the angle between the normal vectors $\vec{n}_1 = \langle 3, -2, 1\rangle$ and $\vec{n}_2 = \langle 2, 1, -3\rangle$:

$\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \dfrac{6-2-3}{\sqrt{9+4+1}\sqrt{4+1+9}} = \dfrac{1}{14}$. So $\theta = \cos^{-1}(1/14) \simeq 85.9°$.

12.5 # 61: The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\vec{u} = \langle -a, b, 0\rangle$ and $\vec{v} = \langle -a, 0, c\rangle$ lie in the plane, and $\vec{n} = \vec{u}\times\vec{v} = \langle bc, ac, ab\rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc$ (plugging in any of the given points to get the right hand side). If a, b, c are nonzero we can divide by abc and rewrite this equation as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

12.5 # 75: $L_1 : x = y = z$, and $L_2 : x + 1 = y/2 = z/3$; so if a point (x, y, z) lies on both L_1 and L_2 then necessarily $y = z$ and $y/2 = z/3$, which implies that $y = z = 0$; however, considering L_1 this implies that $x = 0$, while considering L_2 we get $x + 1 = 0$, a contradiction. So the lines do not intersect. Moreover, the direction vectors are $\vec{v}_1 = \langle 1, 1, 1\rangle$ for L_1 and $\vec{v}_2 = \langle 1, 2, 3\rangle$ for L_2 (the denominators in the symmetric equations). So the lines are not parallel either; hence they are skew lines.

Since L_1 and L_2 are skew, they can be viewed as lying in two parallel planes, and the distance between the skew lines is the same as the distance between these planes. The common normal vector to the planes must be perpendicular to both $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$ (the direction vectors of the two lines). So set $\vec{n} = \langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 3-2, -3+1, 2-1\rangle = \langle 1,-2,1\rangle$. From above, we have found that (0,0,0) and $(-1,0,0)$ are points on L_1 and L_2 respectively. So equations of the planes P_1 containing L_1 and P_2 containing L_2 are respectively $x-2y+z=0$ and $x-2y+z=-1$.

The distance between the parallel planes P_1 and P_2 is the same as the distance of any point on P_2 to P_1; so for example we calculate the distance from the point $(-1,0,0)$ to the plane $x-2y+z=0$. For this we use formula 9 on page 801: $D = \dfrac{|-1-2\cdot 0+0|}{\sqrt{1^2+(-2)^2+1^2}} = \dfrac{1}{\sqrt{6}}$.

Alternative solution: consider the points $Q_1 = (0,0,0)$ on L_1 and $Q_2 = (-1,0,0)$ on L_2: then the distance between the lines is the absolute value of the component of $\overrightarrow{Q_1Q_2} = \langle -1,0,0\rangle$ along the common normal vector $\vec{n} = \langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 1,-2,1\rangle$. So $D = \dfrac{|\overrightarrow{Q_1Q_2}\cdot\vec{n}|}{|\vec{n}|} = \dfrac{1}{\sqrt{6}}$.

Problem 1. a) The line from E to P has the direction of $\overrightarrow{EP} = \langle x_0-2, y_0, z_0\rangle$ and goes through $(2,0,0)$, so it can be parametrized as $\begin{cases} x = 2+(x_0-2)t, \\ y = y_0 t, \\ z = z_0 t. \end{cases}$

It intersects the yz-plane when $x=0$, i.e. $2+(x_0-2)t=0$, which gives $t = \dfrac{2}{2-x_0}$. So $y = \dfrac{2y_0}{2-x_0}$, $z = \dfrac{2z_0}{2-x_0}$.

(We assume $x_0 < 2$ because otherwise the point P would lie behind the observer).

b) The image on the screen of a line segment in space is contained in the intersection of the plane containing E and the line segment with the yz-plane. This intersection is a line, therefore the image is a line segment on the screen.

c) Using (a), $(-1,-3,1)$ is displayed at $(-2,\frac{2}{3})$ in the yz-plane, and $(-2,4,6)$ is displayed at $(2,3)$, so the image on the screen is the line segment from $(-2,\frac{2}{3})$ to $(2,3)$.

d) The trajectory will again be a line segment. The velocity of the bird is $\vec{v} = \overrightarrow{P_0P_1} = \langle -1,7,5\rangle$, so at time t its position is $(-1-t, -3+7t, 1+5t)$. By the formula of part (a), this is displayed at

$$y = \frac{14t-6}{t+3} = \frac{14-\frac{6}{t}}{1+\frac{3}{t}},\quad z = \frac{10t+2}{t+3} = \frac{10+\frac{2}{t}}{1+\frac{3}{t}}.$$

Taking the limit as $t\to\infty$, the position on the screen approaches $(14,10)$ (even though the actual position of the bird in 3D space is further and further away).

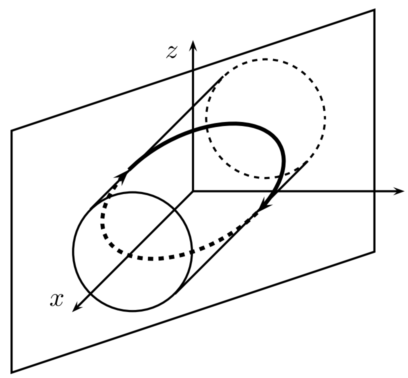
e) The top of the fence is the line $x=1$, $z=1$ (directed along $\hat{\jmath} = \langle 0,1,0\rangle$). The points hidden by the fence are those which lie below the plane that contains E and the top of the fence. This plane also contains the point $R=(1,0,1)$, so it has normal vector $\hat{\jmath}\times\overrightarrow{ER} = \langle 0,1,0\rangle \times \langle -1,0,1\rangle = \langle 1,0,1\rangle$. Therefore its equation is $x+z=2$. More precisely, a point $P:(x,y,z)$ is hidden from the observer if $x<1$

(so it lies further from the observer than the plane of the fence) and $x+z<2$ (so it lies below the plane through E and the top of the fence).

Recall from (d) that the position of the bird at time t is $(-1-t,-3+7t,1+5t)$. The x coordinate is always <1 for $t\geq 0$; and $x+z=4t<2$ when $t<1/2$. So the hidden portion of the trajectory corresponds to $t<1/2$, i.e. from $P_0=(-1,-3,1)$ to $P_{1/2}=(-\frac{3}{2},\frac{1}{2},\frac{7}{2})$.

(An alternative approach is to first find the image of the fence on the screen: using the formula of (a), one can show that the top of the fence is represented by the horizontal line $z=2$ on the screen. The bird is always further from the observer than the fence ($x<1$ for $t\geq 0$), and using the formula found in (d), its image on the screen lies below the top of the fence ($z<2$) precisely when $(10t+2)/(t+3)<2$, or $10t+2<2t+6$, i.e. $t<1/2$. Hence the hidden portion corresponds to $t<1/2$.)

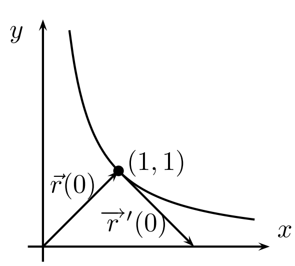
13.1 # 14: Since $x=\cos t$ and $y=-\cos t$, the curve lies in the vertical plane $y=-x$. Moreover, $y^2+z^2=\cos^2 t+\sin^2 t=1$, so the curve lies on the cylinder $y^2+z^2=1$ (or also on the cylinder $x^2+z^2=1$). It is therefore the ellipse where the cylinder and the plane intersect.



13.1 # 22: $x=e^{-t}\cos 10t$, $y=e^{-t}\sin 10t$, $z=e^{-t}$: then $x^2+y^2=e^{-2t}\cos^2 10t+e^{-2t}\sin^2 10t=e^{-2t}=z^2$. So the curve lies on the cone $z^2=x^2+y^2$. Also, z is always positive. So the graph must be I.

13.1 # 23: $x=\cos t$, $y=\sin t$, $z=\sin 5t$: then $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the curve lies on the circular cylinder of radius 1 centered on the z-axis. Each of x,y,z is periodic, and the curve repeats itself after t reaches 2π; moreover z oscillates between -1 and 1 (5 times over the interval $0\leq t\leq 2\pi$). So the graph must be V.

13.2 # 6: (a),(c) $x(t)=e^t$, $y(t)=e^{-t}$; note that eliminating t gives $y=1/x$.



(b) $\overrightarrow{r}'(t)=e^t\hat{\imath}-e^{-t}\hat{\jmath}$.

13.2 # 9: differentiating each component (and using the product rule), $\overrightarrow{r}'(t) = \langle \sin t + t\cos t, 2t, \cos 2t - 2t\sin 2t\rangle$.

13.2 # 25: differentiating $\vec{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, we get
$\overrightarrow{r}'(t) = \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t}\rangle$.
The point $(1,0,1)$ corresponds to $t = 0$, and the tangent vector there is $\overrightarrow{r}'(0) = \langle -1, 1, -1\rangle$. Thus, the tangent line is directed along the vector $\langle -1, 1, -1\rangle$, and parametric equations are $x = 1 - t$, $y = t$, $z = 1 - t$.

13.2 # 31: Note that $\vec{r}_1{}'(t) = \langle 1, 2t, 3t^2\rangle$ while $\vec{r}_2{}'(t) = \langle \cos t, 2\cos 2t, 1\rangle$. Since both curves pass through the origin at $t = 0$, the tangent vectors there are respectively $\overrightarrow{r}'_1(0) = \langle 1, 0, 0\rangle$ and $\overrightarrow{r}'_2(0) = \langle 1, 2, 1\rangle$. The angle θ between these two vectors satisfies $\cos\theta = \dfrac{\langle 1,0,0\rangle \cdot \langle 1,2,1\rangle}{\sqrt{1^2+0^2+0^2}\sqrt{1^2+2^2+1^2}} = \dfrac{1}{\sqrt{6}}$, and $\theta = \cos^{-1}(\frac{1}{\sqrt{6}}) \simeq 66°$.

13.2 # 47: By eq. 5 of Theorem 3, $\frac{d}{dt}(\overrightarrow{r}(t) \times \overrightarrow{r}'(t)) = \overrightarrow{r}'(t) \times \overrightarrow{r}'(t) + \overrightarrow{r}(t) \times \overrightarrow{r}''(t)$. But $\overrightarrow{r}'(t) \times \overrightarrow{r}'(t) = \vec{0}$. So $\frac{d}{dt}(\overrightarrow{r}(t) \times \overrightarrow{r}'(t)) = \overrightarrow{r}(t) \times \overrightarrow{r}''(t)$.

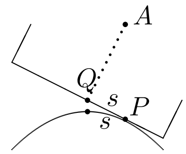
13.2 # 49: $\frac{d}{dt}(|\vec{r}(t)|^2) = \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \overrightarrow{r}'(t) \cdot \overrightarrow{r}(t) + \overrightarrow{r}(t) \cdot \overrightarrow{r}'(t) = 2\overrightarrow{r}(t) \cdot \overrightarrow{r}'(t)$.
However, by the chain rule, $\frac{d}{dt}(|\vec{r}(t)|^2) = 2|\vec{r}(t)|\frac{d}{dt}|\vec{r}(t)|$.
So $\frac{d}{dt}|\vec{r}(t)| = \dfrac{1}{2|\vec{r}(t)|}\frac{d}{dt}(|\vec{r}(t)|^2) = \dfrac{1}{2|\overrightarrow{r}(t)|}2\overrightarrow{r}(t) \cdot \overrightarrow{r}'(t) = \dfrac{1}{|\overrightarrow{r}(t)|}\overrightarrow{r}(t) \cdot \overrightarrow{r}'(t)$.

13.2 # 50: If $\vec{r}(t) \perp \overrightarrow{r}'(t)$, then $\overrightarrow{r}(t) \cdot \overrightarrow{r}'(t) = 0$. So, using the result of # 49, so $\frac{d}{dt}|\vec{r}(t)| = 0$, and $|\vec{r}(t)|$ is constant. (Or: $0 = 2\vec{r}(t) \cdot \vec{r}'(t) = \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{d}{dt}(|\vec{r}(t)|^2)$, so $|\vec{r}(t)|^2$ is constant and hence so is $|\vec{r}(t)|$.) So the trajectory remains at a constant distance from the origin, i.e. it lies on a sphere centered at the origin.

13.4 # 10: Velocity: $\vec{v}(t) = d\vec{r}/dt = \langle -2\sin t, 3, 2\cos t\rangle$.
Acceleration: $\vec{a}(t) = d\vec{v}/dt = \langle -2\cos t, 0, -2\sin t\rangle$.
Speed: $|\vec{v}(t)| = \sqrt{(-2\sin t)^2 + 3^2 + (2\cos t)^2} = \sqrt{4+9} = \sqrt{13}$ (independent of t).

Answers to Problem 2

(a) Let $P : (x(s), y(s))$ be the point where the wheel touches the road, Q the point on the wheel that was initially in contact with the road, and $A : (x_1(s), y_1(s))$ the axle.



The unit tangent vector to the road at P is $\hat{T} = \langle x', y'\rangle$; so $\overrightarrow{PQ} = -s\langle x', y'\rangle$, and $\overrightarrow{QA} = \langle -y', x'\rangle$, hence: $\quad x_1 = x - sx' - y', \quad y_1 = y - sy' + x'$.

(b) $\langle x_1', y_1'\rangle = \dfrac{d}{ds}(\langle x - sx' - y', y - sy' + x'\rangle) = \langle -sx'' - y'', -sy'' + x''\rangle$.
We must have $y_1' = -sy'' + x'' = 0$, so $s = x''/y''$.
(c) Since s is arclength, $|\mathbf{r}'| = 1$, so $\frac{d}{ds}(\mathbf{r}' \cdot \mathbf{r}') = 2\mathbf{r}' \cdot \mathbf{r}'' = 0$.
So $x'x'' + y'y'' = 0$, and $s = x''/y'' = -y'/x' = -(dy/ds)/(dx/ds)$.

(d) $s = \int_0^x \sqrt{1+(dy/dx)^2}\,dx = \int_0^x \sqrt{1+f'(x)^2}\,dx$. (here $f' = df/dx$, not df/ds!).
On the other hand, $f'(x) =$ slope of the tangent to the road $= (dy/ds)/(dx/ds)$.
So $f'(x) = -s = -\int_0^x \sqrt{1+f'(t)^2}\,dt$. Differentiating, $f''(x) = -\sqrt{1+f'(x)^2}$, or $g'(x) = -\sqrt{1+g(x)^2}$.

(e) Recall $\sinh'(x) = \cosh(x) = \sqrt{1+\sinh^2(x)}$. Hence $g(x) = -\sinh(x)$ is a solution; we want $g(0) = 0$ since the road is horizontal at $(0,-1)$. Integrating, $f(x) = -\cosh(x)$ (no constant term since we want $f(0) = -1$).
This formula is valid for $-1 \le s = -f'(x) \le 1$, i.e. $|x| \le \sinh^{-1}(1)$; beyond that position, the corner of the wheel touches the road, and we need to continue with another arch of the same shape $y = -\cosh(x-c)$.