Math 53 Homework 3

Due Tuesday 9/20/11 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 9/12 – Equations of lines and planes

- **Read:** section 12.5.
- **Work:** 12.5: (1), <u>5</u>, (9), (19), (25), <u>33</u>, (37), (43), <u>46</u>, (49), <u>56</u>, <u>61</u>, (63), (69), <u>75</u>.
  <u>Problem 1</u> below.

(Hint for 12.5 # 75: even though we are given symmetric equations rather than parametric ones, this is similar to examples 3 and 10 in the book.)

Wednesday 9/14 – Parametric equations and vector functions

- **Read:** sections 13.1 to middle of p. 820; 13.2 to middle of p. 826.
- **Work:** 13.1: (9), <u>14</u>, (19), <u>22</u>, <u>23</u>, (25), (27), (41).
  13.2: (1), (3), <u>6</u>, <u>9</u>, (19), <u>25</u>, <u>31</u>, (32).

Friday 9/16 – Velocity, acceleration

- **Read:** sections 13.2 (the rest); 13.3 p. 830-831; 13.4 to p. 841.
- **Work:** 13.2: (39), (45), <u>47</u>, <u>49</u>, <u>50</u>.
  13.4: (3), <u>10</u>, (15).
- **Bonus problem** (extra credit, hard): Problem 2 below.

Problem 1. In 3D computer graphics, one needs to represent 3-dimensional objects on a plane screen, by drawing a given point P at the place where the line from P to the eye meets the screen. Suppose that the screen is the yz-plane, and the eye is at $E:(2,0,0)$.

a) At what point $Q:(y,z)$ in the yz-plane should one represent the point $P:(x_0,y_0,z_0)$? (Express y and z in terms of the coordinates of P. Assume that $x_0<2$. Why is this assumption legitimate?)

b) What does the image on the screen of a line segment in space look like? (Justify your answer.)

c) A line segment connects $P_0:(-1,-3,1)$ to $P_1:(-2,4,6)$. What is drawn on the screen?

(continued on next page)

d) A bird leaves from P_0 at time $t = 0$, and flies in a straight line at constant speed in such a way that it passes through P_1 at time $t = 1$.

What does the trajectory of the bird (for $t \geq 0$) look like on the screen? Show that, as t tends to infinity, the trajectory on the screen tends to a limit point (the "vanishing point"), and give its coordinates.

e) In fact, part of the trajectory of the bird is hidden by a vertical fence erected in front of the observer. The fence lies in the plane $x = 1$, and its top is at the altitude $z = 1$. What portion of the trajectory is hidden? (*Hint:* the points hidden by the fence are exactly those which lie below a certain plane passing through E.)

Problem 2 (extra credit, hard)

The purpose of this problem is to determine the shape of the road on which a square wheel rolls smoothly.

Suppose that the square wheel has sidelength 2, and starts with its axle at the origin and its sides parallel to the coordinate axes, touching the road at the point $(0, -1)$. The shape of the road is described by a parametric equation $\mathbf{r}(s) = x(s)\hat{\imath} + y(s)\hat{\jmath}$, using as parameter the arclength s along the road.

a) Given any shape of the road $(x(s), y(s))$, suppose that the square wheel rolls on it without slipping. Find parametric equations for the position $(x_1(s), y_1(s))$ of the axle when the contact point between the road and the wheel is at $(x(s), y(s))$. Express your formulas for $(x_1(s), y_1(s))$ in terms of s, $x = x(s)$, $y = y(s)$ and their derivatives.

b) Now impose the condition that the axle moves on a horizontal trajectory. Compute the velocity of the axle in terms of $\mathbf{r}' = d\mathbf{r}/ds$ and $\mathbf{r}'' = d^2\mathbf{r}/ds^2$, and find a formula for s in terms of $x'' = d^2x/ds^2$ and $y'' = d^2y/ds^2$.

c) Use the fact that s measures arclength to show that $\mathbf{r}'$ and $\mathbf{r}''$ are perpendicular. Deduce from this and from (b) a formula for s in terms of dx/ds and dy/ds only.

d) We now wish to describe the road as the graph of a function $y = f(x)$. Give an integral formula expressing the arclength s as a function of x, and use the result of (c) to show that $g(x) = f'(x)$ is the solution of a differential equation.

e) Find a solution to the differential equation of (d) with $g(0) = 0$ (hint: try hyperbolic trigonometric functions); deduce the shape of the road $y = f(x)$. For which values of x is your formula valid? (what happens afterwards?)