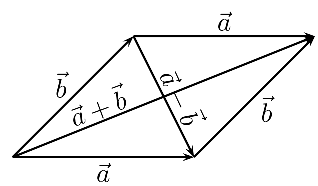
Math 53 Homework 2 – Solutions

12.3 # 19: $\cos\theta = \dfrac{\vec a\cdot\vec b}{|\vec a|\,|\vec b|} = \dfrac{(0)(1)+(1)(2)+(1)(-3)}{\sqrt{0^2+1^2+1^2}\sqrt{1^2+2^2+(-3)^2}} = \dfrac{-1}{\sqrt{2}\sqrt{14}} = \dfrac{-1}{2\sqrt{7}}$, and $\theta = \cos^{-1}(-\frac{1}{2\sqrt7}) \simeq 101^o$.

12.3 # 23: (a) $\vec a\cdot\vec b = (-5)(6)+(3)(-8)+(7)(2) = -40 \neq 0$, so $\vec a$ and $\vec b$ are not orthogonal. Also, $\vec a$ is not a scalar multiple of $\vec b$, so they are not parallel.
(b) $\vec a\cdot\vec b = (4)(-3)+(6)(2) = 0$, so $\vec a$ and $\vec b$ are orthogonal.
(c) $\vec a\cdot\vec b = (-1)(3)+(2)(4)+(5)(-1) = 0$, so $\vec a$ and $\vec b$ are orthogonal.
(d) $\vec a = -\frac{2}{3}\vec b$, so $\vec a$ and $\vec b$ are parallel.

12.3 # 35: $|\vec a| = \sqrt{3^2+(-4)^2} = 5$, so the scalar projection of $\vec b$ onto $\vec a$ is $\mathrm{comp}_{\vec a}(\vec b) = \dfrac{\vec a\cdot\vec b}{|\vec a|} = \dfrac{3\cdot 5+(-4)\cdot 0}{5} = 3$, and the vector projection of $\vec b$ is $\mathrm{proj}_{\vec a}(\vec b) = \left(\dfrac{\vec a\cdot\vec b}{|\vec a|}\right)\dfrac{\vec a}{|\vec a|} = 3\cdot\frac15\langle 3,-4\rangle = \langle \frac95, -\frac{12}{5}\rangle$.

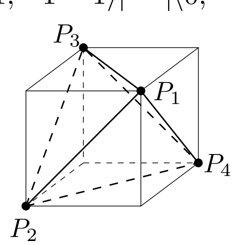
12.3 # 56: By assumption, the quadrilateral is a parallelogram, so its opposite sides are represented by the same vectors:



By assumption, $|\vec a| = |\vec b|$. The diagonals correspond to the vectors $\vec a+\vec b$ and $\vec a-\vec b$; we compute $(\vec a+\vec b)\cdot(\vec a-\vec b) = \vec a\cdot\vec a - \vec b\cdot\vec b = |\vec a|^2 - |\vec b|^2 = 0$. Hence the diagonals are perpendicular.

12.3 # 59: (a) $\vec a+\vec b$ and $\vec a-\vec b$ represent the diagonals of a parallelogram with sides $\vec a$ and $\vec b$ (see figure above).
So the identity $|\vec a+\vec b|^2 + |\vec a-\vec b|^2 = 2|\vec a|^2+2|\vec b|^2$ states that the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of its four sides.
(b) $|\vec a+\vec b|^2+|\vec a-\vec b|^2 = (\vec a+\vec b)\cdot(\vec a+\vec b)+(\vec a-\vec b)\cdot(\vec a-\vec b) =$
$= (\vec a\cdot\vec a+2\vec a\cdot\vec b+\vec b\cdot\vec b)+(\vec a\cdot\vec a-2\vec a\cdot\vec b+\vec b\cdot\vec b) = 2\vec a\cdot\vec a+2\vec b\cdot\vec b = 2|\vec a|^2+2|\vec b|^2$.

Problem 1. (a) $P_1=(1,1,1)$, $P_2=(1,-1,-1)$, $P_3=(-1,-1,1)$, $P_4=(-1,1,-1)$. Each pair of points differs by two sign changes from the others. All six edges of the tetrahedron are diagonals of faces of the cube and hence have the same length. For instance, $|\overrightarrow{P_1P_2}| = |\langle 1-1,-1-1,-1-1\rangle| = |\langle 0,-2,-2\rangle| = 2\sqrt2$.



(b) Adjacent edges: $\overrightarrow{P_1P_2} = \langle 0, -2, -2\rangle$, $\overrightarrow{P_1P_3} = \langle -2, -2, 0\rangle$.

$$\cos\alpha = \frac{\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1P_3}}{|\overrightarrow{P_1P_2}||\overrightarrow{P_1P_3}|} = \frac{4}{(2\sqrt{2})^2} = \frac{1}{2}; \quad \alpha = \pi/3 = 60^\circ$$

The faces are equilateral triangles, so the angles are 60°.
Opposite edges: $\overrightarrow{P_1P_2} = \langle 0, -2, -2\rangle$, $\overrightarrow{P_3P_4} = \langle 0, 2, -2\rangle$.

$$\cos\beta = \frac{\overrightarrow{P_1P_2}\cdot\overrightarrow{P_3P_4}}{|\overrightarrow{P_1P_2}||\overrightarrow{P_3P_4}|} = \frac{0}{(2\sqrt{2})^2} = 0; \quad \beta = \pi/2 = 90^\circ$$

By symmetry the perpendicular bisector to an edge contains the opposite edge, so these two edges are perpendicular to each other.
(c) $\cos\theta = \overrightarrow{OP_1}\cdot\overrightarrow{OP_2}/|\overrightarrow{OP_1}||\overrightarrow{OP_2}| = \langle 1,1,1\rangle\cdot\langle 1,-1,-1\rangle/(\sqrt{3})^2 = -1/3$. $\theta \approx 1.91$ radians (109.5°).

12.4 # 16: (a) $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta = 3\cdot 2\cdot\sin\frac{\pi}{2} = 6$.
(b) $\vec{a}\times\vec{b}$ is orthogonal to $\vec{b}$, so it lies in the xy-plane and its z-component is 0. By the right-hand rule, its y-component is negative and its x-component is positive.

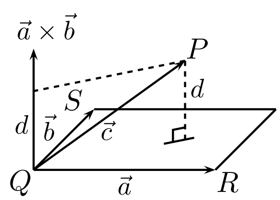
12.4 # 31: (a) The plane through P, Q, R contains the vectors $\overrightarrow{PQ} = \langle 4, 3, -2\rangle$ and $\overrightarrow{PR} = \langle 5, 5, 1\rangle$, so a vector orthogonal to the plane is

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 3 & -2 \\ 5 & 5 & 1 \end{vmatrix} = \langle 3\cdot 1-(-2)\cdot 5, (-2)\cdot 5-4\cdot 1, 4\cdot 5-3\cdot 5\rangle = \langle 13, -14, 5\rangle$$

(or any multiple thereof).
(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $|\overrightarrow{PQ}\times\overrightarrow{PR}| = |\langle 13, -14, 5\rangle| = \sqrt{13^2+(-14)^2+5^2} = \sqrt{390}$, so the area of the triangle is $\frac{1}{2}\sqrt{390}$.

12.4 # 35: $\det(\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{PS}) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} =$ $2-3-2=-3$; so the volume of the parallelepiped is 3.

12.4 # 44: (a)



Up to sign, the distance d from P to the plane is the component of $\vec{c} = \overrightarrow{QP}$ along the direction perpendicular to the plane (see picture). Since $\vec{a} = \overrightarrow{QR}$ and $\vec{b} = \overrightarrow{QS}$ lie in the plane, $\vec{a}\times\vec{b}$ is perpendicular to the plane. So the answer is (up to sign)

$$\mathrm{comp}_{\vec{a}\times\vec{b}}(\vec{c}) = \frac{(\vec{a}\times\vec{b})\cdot\vec{c}}{|\vec{a}\times\vec{b}|}.$$

This quantity may be positive or negative depending on which side of the plane P lies on, whereas distance is always measured positively, i.e. $d = |\mathrm{comp}_{\vec{a}\times\vec{b}}(\vec{c})|$.

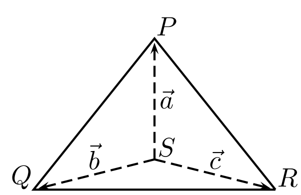
Alternative solution: d is the height of the parallelepiped with edges $\vec{a}$, $\vec{b}$ and $\vec{c}$. However, since the volume is the area of the base times the height, we get that d = (volume)/(area of base). The volume is given by the absolute value of the determinant or triple product $(\vec{a}\times\vec{b})\cdot\vec{c}$. The area of the base (a parallelogram with sides $\vec{a}$ and $\vec{b}$) is given by $|\vec{a}\times\vec{b}|$. Dividing, we get the desired formula for d.
(b) $\vec{a}=\overrightarrow{QR}=\langle -1,2,0\rangle$, $\vec{b}=\overrightarrow{QS}=\langle -1,0,3\rangle$, and $\vec{c}=\overrightarrow{QP}=\langle 1,1,4\rangle$. Then

$$\vec{a}\times\vec{b}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \langle 6,3,2\rangle,$$

and $(\vec{a}\times\vec{b})\cdot\vec{c}=\langle 6,3,2\rangle\cdot\langle 1,1,4\rangle=17$, so $d=\dfrac{|(\vec{a}\times\vec{b})\cdot\vec{c}|}{|\vec{a}\times\vec{b}|}=\dfrac{17}{\sqrt{36+9+4}}=\dfrac{17}{7}$.

12.4 # 49: (a) No. $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ if and only if $\vec{a}\cdot(\vec{b}-\vec{c})=0$, which happens precisely when $\vec{a}$ is perpendicular to $\vec{b}-\vec{c}$; this can occur even if $\vec{b}\neq\vec{c}$.
(b) No. $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}\Leftrightarrow\vec{a}\times(\vec{b}-\vec{c})=0$, which means that $\vec{a}$ is parallel to $\vec{b}-\vec{c}$; this can happen with $\vec{b}\neq\vec{c}$.
(c) Yes. If $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ then $\vec{a}$ is perpendicular to $\vec{b}-\vec{c}$ by part (a). From part (b), if $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ then $\vec{a}$ is also parallel to $\vec{b}-\vec{c}$. Since $\vec{a}\neq\vec{0}$ is both parallel and perpendicular to $\vec{b}-\vec{c}$, we must have $\vec{b}-\vec{c}=0$, so $\vec{b}=\vec{c}$.

p. 794 part 1:



The vector coming out of the face opposite P (the bottom face) is $\vec{v}_1=\frac{1}{2}\overrightarrow{SR}\times\overrightarrow{SQ}=\frac{1}{2}\vec{c}\times\vec{b}$. (Indeed, this vector is perpendicular to the face, its magnitude is equal to the area of the triangle, and by the right-hand rule it points downwards).
Similarly, for the face opposite Q we have $\vec{v}_2=\frac{1}{2}\overrightarrow{SP}\times\overrightarrow{SR}=\frac{1}{2}\vec{a}\times\vec{c}$, and for the face opposite R we have $\vec{v}_3=\frac{1}{2}\overrightarrow{SQ}\times\overrightarrow{SP}=\frac{1}{2}\vec{b}\times\vec{a}$.
Finally, for the face opposite S (the front face), $\vec{v}_4=\frac{1}{2}\overrightarrow{PQ}\times\overrightarrow{PR}=\frac{1}{2}(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$.
Expanding and using the properties of cross product, we get:
$\vec{v}_4=\frac{1}{2}(\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a})=\frac{1}{2}(-\vec{c}\times\vec{b}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{0})=-\vec{v}_1-\vec{v}_3-\vec{v}_2$.
So $\vec{v}_1+\vec{v}_2+\vec{v}_3+\vec{v}_4=\vec{0}$.

part 3: since the three edges meeting at the vertex S are mutually perpendicular, we can set up our coordinate system so that S is at the origin, SQ is the x-axis, SR is the y-axis, SP is the z-axis. Then the face opposite P lies in the xy-plane and has area A, so $\vec{v}_1=\langle 0,0,-A\rangle$. Similarly, the faces opposite Q and R lie in the yz and xz planes, and have areas B and C, so $\vec{v}_2=\langle -B,0,0\rangle$ and $\vec{v}_3=\langle 0,-C,0\rangle$. Using the result of part 1, we deduce that $\vec{v}_4=-\vec{v}_1-\vec{v}_2-\vec{v}_3=\langle B,C,A\rangle$. Hence the area of the fourth face is $D=|\vec{v}_4|=\sqrt{B^2+C^2+A^2}$, so $D^2=B^2+C^2+A^2$.