Math 53 Homework 1 – Solutions

(These solutions are not necessarily complete; in particular some diagrams have been omitted for convenience and brevity.)

10.1 # 11: (a) $x = \sin\theta$, $y = \cos\theta$, so $x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1$.

(b) right half of the unit circle, traced clockwise from $(\sin 0, \cos 0) = (0, 1)$ to $(\sin\pi, \cos\pi) = (0, -1)$.

10.1 # 13: (a) $x = \sin t$, $y = 1/\sin t$, so $xy = 1$.

(b) as t increases from 0 to $\pi/2$, x increases from 0 to 1; hence, we get the portion of the hyperbola $y = 1/x$ for $0 < x < 1$, traced from left to right (downwards).

10.1 # 31: (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \le t \le 1$: the curve clearly passes through $P_1\,(x_1, y_1)$ at $t = 0$ and through $P_2\,(x_2, y_2)$ at $t = 1$. Since x and y each vary at a constant rate, the trajectory is along a straight line. (Or, more explicitly, we can eliminate t: $t = \frac{x-x_1}{x_2-x_1}$, so $y = y_1 + \frac{y_2-y_1}{x_2-x_1}(x - x_1)$, which is indeed a line). Moreover, because the range of values of x for $0 \le t \le 1$ is precisely from x_1 to x_2 (or similarly for y), the trajectory is the line segment from P_1 to P_2.

(b) $x = -2 + (3 - (-2))t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$.

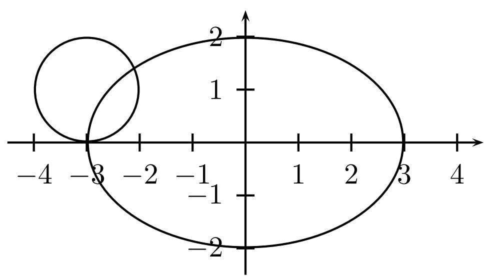
10.1 # 33: the circle of radius 2 centered at $(0, 1)$ can be parametrized by $x = 2\cos t$, $y = 1 + 2\sin t$ where, as t varies from 0 to 2π, the trajectory goes around the circle counterclockwise, starting at $(2, 1)$, and hitting $(0, 3)$ at $t = \pi/2$. Hence:

(a) to get a clockwise orientation, we should change t to $-t$; this yields $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$ (or any interval between consecutive multiples of 2π).

(b) $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 6\pi$.

(c) $x = 2\cos t$, $y = 1 + 2\sin t$, $\pi/2 \le t \le 3\pi/2$.

10.1 # 45: (a) There are two intersection points, one at $(-3, 0)$ and the other near $(-2.1, 1.4)$.



(b) the intersection at $(-3, 0)$ is a collision point, since it is hit by the first particle at $t = 3\pi/2$ and by the second one at the same time $t = 3\pi/2$. On the other hand, the intersection near $(-2.1, 1.4)$ is hit by the first particle (which moves clockwise on the large ellipse) at some time t_1 with $3\pi/2 < t_1 < 2\pi$; while the second particle passes through it for some time t_2 with $0 < t_2 < \pi/2$; since the particles are never there at the same time, it is not a collision point.

Or, more systematically: a collision point corresponds to t such that $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$, i.e. $3\sin t = -3 + \cos t$ and $2\cos t = 1 + \sin t$. From the first equation we get that $\cos t = 3 + 3\sin t$, and plugging into the second equation we obtain $5 + 5\sin t = 0$; this yields $\sin t = -1$, which corresponds to $t = 3\pi/2$, indeed a solution of both equations.

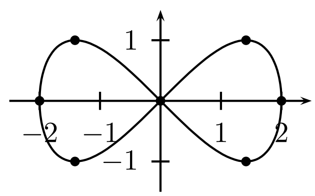
(c) the circle is now centered at $(3, 1)$ instead of $(-3, 1)$. There are still two intersections, at $(3, 0)$ and near $(2.1, 1.4)$; but there are no collision points (for instance because the equations $3\sin t = 3 + \cos t$ and $2\cos t = 1 + \sin t$ imply that $\sin t = 7/5$ and $\cos t = 6/5$, impossible.)

10.2 # 7: (a) $x = 1 + \ln t$, $y = t^2 + 2$: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$. The point $(1, 3)$ is hit at $t = 1$, so $dy/dx = 2$, and the tangent is $y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(b) $t = e^{x-1}$, so $y = e^{2x-2} + 2$, and $dy/dx = 2e^{2x-2}$; when $x = 1$ we have $dy/dx = 2e^0 = 2$, and the equation of the tangent follows as in (a).

10.2 # 19: $x = 2\cos\theta$, $y = \sin 2\theta$: to find horizontal tangents, we compute $dy/d\theta = 2\cos 2\theta$, so $dy/d\theta = 0$ if and only if $2\theta = \frac{\pi}{2} + n\pi$ (n integer), i.e. $\theta = \frac{\pi}{4} + n\frac{\pi}{2}$. This corresponds to the four points $(x, y) = (\pm\sqrt{2}, \pm 1)$ where the curve has horizontal tangencies.

Similarly, for vertical tangents, we find θ for which $dx/d\theta = -2\sin\theta = 0$, i.e. $\theta = n\pi$ (n integer), which gives the two points $(\pm 2, 0)$.



10.2 # 33: The curve lies above the x-axis for $0 \le t \le 1$, and x increases with t, so the area is $\int_2^{e+1} y\,dx = \int_0^1 (t - t^2)\,e^t\,dt$. Using integration by parts twice, we find

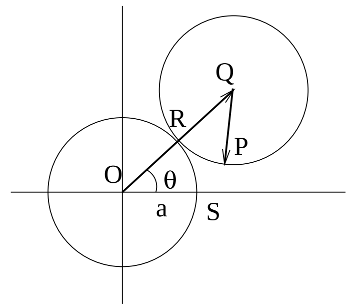
$$\int_0^1 (t - t^2)\,e^t\,dt = (t - t^2)\,e^t\Big|_0^1 - \int_0^1 (1 - 2t)\,e^t\,dt =$$
$$= \int_0^1 (2t - 1)e^t\,dt = (2t - 1)e^t\Big|_0^1 - \int_0^1 2e^t\,dt = (2t - 3)e^t\Big|_0^1 = -e + 3.$$

10.2 # 41: $dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$, and thus $L = \int_0^1 \sqrt{36t^2 + 36t^4}\,dt = \int_0^1 6t\sqrt{1 + t^2}\,dt$. Substituting $u = 1 + t^2$,

$$L = \int_0^1 6t\sqrt{1 + t^2}\,dt = \int_1^2 3\sqrt{u}\,du = 2u^{3/2}\Big|_1^2 = 2(2\sqrt{2} - 1).$$

10.2 # 73: The coordinates of T (see figure in the book) are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from the arc TA, TP has length $r\theta$. Also, TP is perpendicular to OT, so makes an angle $\theta - \frac{\pi}{2}$ with the x-axis (or θ with the negative y axis). So P has coordinates $x = r\cos\theta + r\theta\,\cos(\theta - \frac{\pi}{2}) = r\cos\theta + r\theta\sin\theta$ and $y = r\sin\theta + r\theta\,\sin(\theta - \frac{\pi}{2}) = r\sin\theta - r\theta\cos\theta$.

Problem 1. (a) Let Q be the center of the rolling circle: since $|OQ| = 2a$ and the line OQ passes through the contact point, the coordinates of Q are $(2a\cos\theta, 2a\sin\theta)$.



To find the position of P, observe that $\angle OQP = \theta$: since one circle rolls on the other the arc lengths from S to R and from P to R are equal. So the line QP makes an angle of 2θ with the negative x-axis, and $|QP| = a$. So the coordinates of P are:

$$x = 2a\cos\theta - a\cos 2\theta, \quad y = 2a\sin\theta - a\sin 2\theta.$$

(b) $dx/d\theta = 2a(-\sin\theta + \sin 2\theta)$, and $dy/d\theta = 2a(\cos\theta - \cos 2\theta)$, so

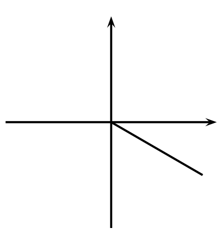
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4a^2(\sin^2\theta + \sin^2 2\theta - 2\sin\theta\sin 2\theta + \cos^2\theta + \cos^2 2\theta - 2\cos\theta\cos 2\theta)$$

which simplifies (using $\sin^2 + \cos^2 = 1$) to $4a^2(2 - 2\sin\theta\sin 2\theta - 2\cos\theta\cos 2\theta) = 4a^2(2 - 2\cos(2\theta - \theta)) = 4a^2(2 - 2\cos\theta)$. So $L = \int_0^{2\pi} 2a\sqrt{2 - 2\cos\theta}\,d\theta$.
Recall that $\cos\theta = 1 - 2\sin^2(\theta/2)$, so $2 - 2\cos\theta = 4\sin^2(\theta/2)$,
and $L = \int_0^{2\pi} 4a\sin(\theta/2)\,d\theta = -8a\cos(\theta/2)|_0^{2\pi} = -8a((-1) - 1) = 16a$.

10.3 # 17: $r = 3\sin\theta \Rightarrow r^2 = 3r\sin\theta \Leftrightarrow x^2 + y^2 = 3y$, i.e. $x^2 + (y - \frac{3}{2})^2 = (\frac{3}{2})^2$, circle of radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$. (Note: multiplying both sides by r in the first step adds the extra solution $r = 0$, which corresponds to the origin; however the origin already lies on the circle, so this is of no consequence).

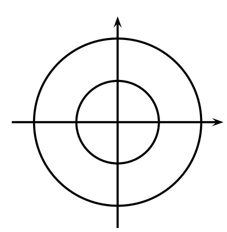
10.3 # 19: $r = \csc\theta \Leftrightarrow r\sin\theta = 1 \Leftrightarrow y = 1$ (horizontal line through $(0, 1)$).

10.3 # 29: $\theta = -\pi/6$:

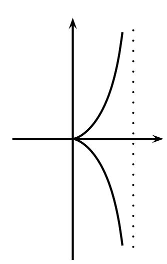


(or the entire line if we allow $r < 0$)

10.3 # 30: $r^2 - 3r + 2 = 0 \Leftrightarrow r = 1$ or $r = 2$:



10.3 # 53: To show that $x = 1$ is an asymptote we must prove that $x \to 1$ as $r \to \infty$. $r \to \infty$ corresponds to $\sin\theta\tan\theta \to \infty$, i.e. $\theta \to \pm\pi/2$. (Note: $r(\theta + \pi) = -r(\theta)$, so if we allow all values of θ we trace the curve twice; instead we restrict ourselves to $-\pi/2 < \theta < \pi/2$, for which $r \geq 0$). Now, $x = r\cos\theta = (\sin\theta\tan\theta)\cos\theta = \sin^2\theta$, which does tend to 1 as $\theta \to \pm\pi/2$. (Note: $\theta \to (\pi/2)^-$ corresponds to $x \to 1$, $y \to +\infty$, while $\theta \to (-\pi/2)^+$ correspond to $x \to 1$, $y \to -\infty$).
Since $x = \sin^2\theta$ takes values ranging between 0 and 1, the curve is contained in the strip $0 \leq x \leq 1$ (and $x = 1$ is never reached). And, since $r(-\theta) = r(\theta)$, the curve is symmetric about the x-axis.



10.3 # 63: $r = 3\cos\theta$ gives: $x = r\cos\theta = 3\cos^2\theta$, $y = r\sin\theta = 3\cos\theta\sin\theta$. Horizontal tangencies occur when $dy/d\theta = -3\sin^2\theta + 3\cos^2\theta = \frac{3}{2}\cos 2\theta = 0$, so for $2\theta = \frac{\pi}{2} + n\pi$ or $\theta = \frac{\pi}{4} + n\frac{\pi}{2}$. Vertical tangencies occur when $dx/d\theta = -6\sin\theta\cos\theta = -3\sin 2\theta = 0$, so for $2\theta = n\pi$ or $\theta = n\frac{\pi}{2}$.

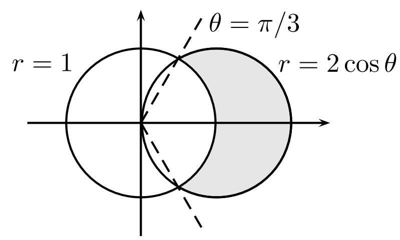
We only need to consider θ between $-\pi/2$ and $\pi/2$ (where $r \geq 0$), the range $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ traces the same curve again ($r(\theta+\pi) = -r(\theta)$). Hence: the tangent is horizontal for $\theta = \pm\frac{\pi}{4}$ (recalling that the curve is the circle of radius $\frac{3}{2}$ centered at $(\frac{3}{2}, 0)$, these are the top and bottom points $(\frac{3}{2}, \pm\frac{3}{2})$); and vertical for $\theta = 0$ (the rightmost point $(3, 0)$) and $\theta = \pm\pi/2$ (at the origin).

10.4 # 7: $A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(4 + 3\sin\theta)^2\, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(16 + 24\sin\theta + 9\sin^2\theta)\, d\theta$.

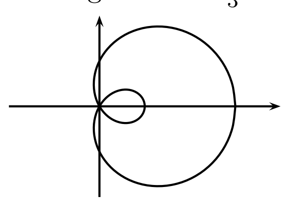
Using parity, the portions of the integral from $-\pi/2$ to 0 and 0 to $\pi/2$ cancel out for $\sin\theta$, while they are equal for the other terms of the integrand; so

$A = \int_0^{\pi/2}(16 + 9\sin^2\theta)\, d\theta = \int_0^{\pi/2}(16 + \frac{9}{2}(1 - \cos 2\theta))\, d\theta = \left[\frac{41}{2}\theta - \frac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{41\pi}{4}$.

10.4 # 23: Inside $r = 2\cos\theta$, outside $r = 1$: the curves intersect when $2\cos\theta = 1$, $\theta = \cos^{-1}(\frac{1}{2}) = \pm\pi/3$. We subtract the unshaded area from the shaded area:
$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(2\cos\theta)^2\, d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}\, d\theta = \int_{-\pi/3}^{\pi/3}(2\cos^2\theta - \frac{1}{2})\, d\theta = \int_{-\pi/3}^{\pi/3}(\cos 2\theta + \frac{1}{2})\, d\theta$
$= \left[\frac{1}{2}\sin 2\theta + \frac{1}{2}\theta\right]_{-\pi/3}^{\pi/3} = (\frac{\sqrt{3}}{4} + \frac{\pi}{6}) - (-\frac{\sqrt{3}}{4} - \frac{\pi}{6}) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$.



10.4 # 35: $r = \frac{1}{2} + \cos\theta$ is positive for $\cos\theta > -\frac{1}{2}$, i.e. $-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$; this corresponds to the large loop; r is negative for $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, the small loop.



The area in the large loop is $2\int_0^{2\pi/3} \frac{1}{2}(\frac{1}{2} + \cos\theta)^2\, d\theta = \int_0^{2\pi/3}(\frac{1}{4} + \cos\theta + \cos^2\theta)\, d\theta = \int_0^{2\pi/3}(\frac{3}{4} + \cos\theta + \frac{1}{2}\cos 2\theta)\, d\theta = \left[\frac{3}{4}\theta + \sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi/3} = (\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{1}{4}\frac{\sqrt{3}}{2}) - 0 = \frac{\pi}{2} + \frac{3\sqrt{3}}{8}$.

Similarly, the area in the small loop is
$2\int_{2\pi/3}^{\pi} \frac{1}{2}(\frac{1}{2} + \cos\theta)^2\, d\theta = \left[\frac{3}{4}\theta + \sin\theta + \frac{1}{4}\sin 2\theta\right]_{2\pi/3}^{\pi} = (\frac{3\pi}{4}) - (\frac{\pi}{2} + \frac{3\sqrt{3}}{8}) = \frac{\pi}{4} - \frac{3\sqrt{3}}{8}$.

Subtracting, the desired area is $(\frac{\pi}{2} + \frac{3\sqrt{3}}{8}) - (\frac{\pi}{4} - \frac{3\sqrt{3}}{8}) = \frac{\pi}{4} + \frac{3\sqrt{3}}{4}$.

10.4 # 45: using equation (5) on p.652,
$L = \int_0^{\pi/3} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{\pi/3} \sqrt{(3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta = \int_0^{\pi/3} 3\, d\theta = \pi$.

12.1 # 13: The radius of the sphere is the distance between (4,3,-1) and (3,8,1), namely $r = \sqrt{(3-4)^2 + (8-3)^2 + (1-(-1))^2} = \sqrt{30}$. Hence, an equation of the sphere is $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

12.1 # 31: $x^2 + z^2 \leq 9$: a cylinder of radius 3 centered on the y-axis. (The intersection of this solid in the xz-plane is the disk $x^2 + z^2 \leq 9$ of radius 3 centered at the origin; since the equation does not involve y, it intersects every plane parallel to the xz-plane in the same manner).

12.1 # 39: $P\,(x, y, z)$ satisfies $|AP| = |BP|$ if and only if

$$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2}.$$

Squaring both sides and expanding, we get:

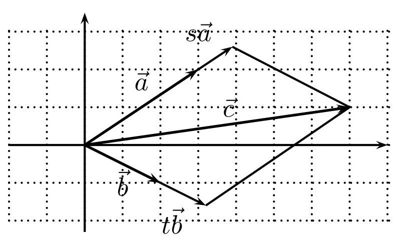
$x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = x^2 - 12x + 36 + y^2 - 4y + 4 + z^4 + 4z + 4$, which simplifies to $14x - 6y - 10z = 9$. This is a plane – in fact, the plane perpendicular to the line segment AB through its midpoint, for symmetry reasons.

12.2 # 29: The two forces are given by the vectors $\vec{F_1} = \langle -300, 0 \rangle$ and $\vec{F_2} = \langle 200 \cos 60^o, 200 \sin 60^o \rangle = \langle 100, 100\sqrt{3} \rangle$. The resultant force is $\vec{F} = \vec{F_1} + \vec{F_2} = \langle -300 + 100, 100\sqrt{3} \rangle = \langle -200, 100\sqrt{3} \rangle$.

Its magnitude is $|\vec{F}| = \sqrt{(-200)^2 + (100\sqrt{3})^2} = 100\sqrt{4+3} = 100\sqrt{7} \simeq 264.6$ N.

The angle with the positive x-axis is determined by $\tan\theta = (100\sqrt{3})/(-200) = -\frac{\sqrt{3}}{2}$. $\tan^{-1}(-\sqrt{3}/2) \simeq -0.714$ radians (or -40.9^o). However, the vector points into the upper-left quadrant, so we must add π, and the angle is $\simeq 3.855$ radians or 139.1^o.

12.2 # 39: (a),(b)



(c) from the sketch, we estimate that $s \simeq 1.3$ and $t \simeq 1.6$.

(d) $s\vec{a} + t\vec{b} = \langle 3s + 2t, 2s - t \rangle$, so $s\vec{a} + t\vec{b} \Leftrightarrow 3s + 2t = 7$ and $2s - t = 1$. Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

12.2 # 45: Consider the triangle ABC, and let D and E be the midpoints of AB and AC. Then $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$, and $\overrightarrow{DE} = \overrightarrow{AE} - \overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC} - \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BC}$. Therefore $\overrightarrow{BC}$ and $\overrightarrow{DE}$ are parallel, and $|\overrightarrow{DE}| = \frac{1}{2}|\overrightarrow{BC}|$.