Math 53 Homework 13 – Solutions

16.9 # 17: Let S_1 be the disk $x^2+y^2 \le 1$ in the xy-plane, oriented downwards. Its normal vector is $\hat{\text{n}} = -\hat{\text{k}}$, so $\vec{F}\cdot\hat{\text{n}} = -\vec{F}\cdot\hat{\text{k}} = -(x^2z+y^2) = -y^2$ (since $z=0$ on S_1). Hence

$$\iint_{S_1} \vec{F}\cdot\hat{\text{n}}\,dS = \iint_{S_1} -y^2\,dS = -\int_0^{2\pi}\int_0^1 (r\sin\theta)^2\,r\,dr\,d\theta.$$

Inner: $\left[\frac{1}{4}r^4\sin^2\theta\right]_0^1 = \frac{1}{4}\sin^2\theta$. Outer: $-\int_0^{2\pi}\frac{1}{4}\sin^2\theta\,d\theta = -\int_0^{2\pi}\frac{1}{8}(1-\cos 2\theta)\,d\theta = -\frac{\pi}{4}$.
Now, we apply the divergence theorem to the closed surface $S\cup S_1$. Observing that $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}(\frac{1}{3}y^3+\tan z) + \frac{\partial}{\partial z}(x^2z+y^2) = z^2+y^2+x^2$, we have:

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1}\vec{F}\cdot d\vec{S} = \iiint_E x^2+y^2+z^2\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= 2\pi\left(\int_0^{\pi/2}\sin\phi\,d\phi\right)\left(\int_0^1\rho^4\,d\rho\right) = (2\pi)(1)(\tfrac{1}{5}) = \tfrac{2}{5}\pi.$$

Finally, $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV - \iint_{S_1}\vec{F}\cdot d\vec{S} = \frac{2}{5}\pi - (-\frac{1}{4}\pi) = \frac{13}{20}\pi$.

16.9 # 19: The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 (or: the net flux out of a small disk around P_1 is negative), and so $\operatorname{div}\vec{F}$ is negative at P_1. Conversely, the vectors that end near P_2 are longer than those that start near P_2, so the net flow is outward near P_2 (or: the net flux out of a small disk around P_2 is positive), and so $\operatorname{div}\vec{F}$ is positive at P_2.

16.9 # 27: $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}(\operatorname{curl}\vec{F})\,dV$, however by Theorem 11 in 16.5 we have $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ (see p. 1065 for the proof), so $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iiint_E 0\,dV = 0$.

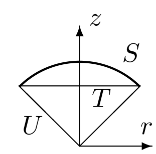
16.8 # 3: The boundary curve C is the circle $x^2+y^2=4$, $z=4$ oriented counterclockwise, so it can be parametrized by $x=2\cos t$, $y=2\sin t$, $z=4$, $0\le t\le 2\pi$. Thus

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_C x^2z^2\,dx + y^2z^2\,dy + xyz\,dz =$$
$$= \int_0^{2\pi} 16(2\cos t)^2(-2\sin t)\,dt + 16(2\sin t)^2(2\cos t)\,dt + 0\,dt = \left[\frac{128}{3}\cos^3 t + \frac{128}{3}\sin^3 t\right]_0^{2\pi} = 0.$$

16.8 # 9: $\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\text{k}} \\ \partial_x & \partial_y & \partial_z \\ yz & 2xz & e^{xy} \end{vmatrix} = (xe^{xy}-2x)\hat{\imath} - (ye^{xy}-y)\hat{\jmath} + (2z-z)\hat{\text{k}}.$

We take S to be the disk $x^2+y^2\le 16$, $z=5$. Since C is oriented counterclockwise (from above), we orient S upward. Then $\hat{\text{n}} = \hat{\text{k}}$, and $\operatorname{curl}\vec{F}\cdot\hat{\text{n}} = z$ on S, where $z=5$. Thus

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{\text{n}}\,dS = \iint_S z\,dS = \iint_S 5\,dS = 5\,\text{area}(S) = 80\pi.$$



Problem 1. Radius of disk T: intersection of $z^2 = x^2+y^2$ (cone) and $x^2+y^2+z^2=2$ (sphere). By elimination, we get $x^2+y^2=z^2=1$, i.e. $z=1$ and radius $r=1$.

a) $\vec{F} = x\hat{\imath}+y\hat{\jmath}$ is horizontal and points radially outwards (away from the z-axis). Therefore, the flux across S is positive ($\vec{F}$ points out of the sphere); the flux across T is zero ($\vec{F}$ is parallel to the horizontal plane containing T); the flux across U is negative ($\vec{F}$ points out of the cone, while the normal vector points up and into the cone).

b) Across S (spherical cap, $\rho = \sqrt{2}$, $\phi < \frac{\pi}{4}$): $dS = \rho^2 \sin\phi \, d\phi \, d\theta = 2\sin\phi \, d\phi \, d\theta$.
Unit normal $\hat{\mathrm{n}} = \frac{1}{\rho}(x\hat{\imath} + y\hat{\jmath} + z\hat{\mathrm{k}})$, hence $\vec{F} \cdot \hat{\mathrm{n}} = \frac{1}{\rho}(x^2 + y^2) = \rho \sin^2\phi = \sqrt{2}\sin^2\phi$, and

$$\iint_S \vec{F} \cdot \hat{\mathrm{n}} \, dS = \int_0^{2\pi} \int_0^{\pi/4} (\sqrt{2}\sin^2\phi)\,(2\sin\phi)\, d\phi\, d\theta = 2\sqrt{2}\,(2\pi) \int_0^{\pi/4} \sin^3\phi \, d\phi$$
$$= 4\pi\sqrt{2} \int_0^{\pi/4} \sin\phi(1 - \cos^2\phi)\, d\phi = 4\pi\sqrt{2} \left[-\cos\phi + \frac{1}{3}\cos^3\phi \right]_0^{\pi/4}$$
$$= 4\pi\sqrt{2} \Big[(-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}}) - (-1 + \frac{1}{3}) \Big] = \frac{(8\sqrt{2} - 10)\pi}{3}.$$

Across T: $\hat{\mathrm{n}} = \hat{\mathrm{k}}$, so $\vec{F} \cdot \hat{\mathrm{n}} = 0$ and $\displaystyle\iint_T \vec{F} \cdot \hat{\mathrm{n}} \, dS = 0$.
Across U (cone, graph of $f(x, y) = \sqrt{x^2 + y^2}$ over unit disk): $\hat{\mathrm{n}} \, dS = \langle -f_x, -f_y, 1 \rangle \, dA = \langle -x/r, -y/r, 1 \rangle \, dA$, so $\vec{F} \cdot \hat{\mathrm{n}} \, dS = \langle x, y, 0 \rangle \cdot \langle -x/r, -y/r, 1 \rangle \, dA = (-r)\,(r \, dr \, d\theta)$.

$$\iint_U \vec{F} \cdot \hat{\mathrm{n}} \, dS = \int_0^{2\pi} \int_0^1 -r^2 \, dr \, d\theta = -(2\pi)\,\frac{1}{3} = -\frac{2\pi}{3}.$$

c) $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0) = 2$. Therefore, the flux out of the solid cone D_1 is

$$\iint \vec{F} \cdot \hat{\mathrm{n}} \, dS = \iiint_{D_1} 2 \, dV = 2 \operatorname{volume}(D_1) = 2\,(\frac{1}{3}\,\pi) = \frac{2\pi}{3}. \qquad \text{(volume} = \text{base} \times \text{height}\,/3).$$

Flux out of the region D_2 bounded by S and U:

$$\iint \vec{F} \cdot \hat{\mathrm{n}} \, dS = \iiint_{D_2} 2 \, dV = 2 \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= 2\,(2\pi) \int_0^{\pi/4} \frac{1}{3}\rho^3 \sin\phi \Big|_0^{\sqrt{2}} \, d\phi$$
$$= 4\pi \, \frac{2\sqrt{2}}{3} \int_0^{\pi/4} \sin\phi \, d\phi = \frac{8\pi\sqrt{2}}{3}(-\frac{1}{\sqrt{2}} - (-1)) = \frac{8(\sqrt{2} - 1)\pi}{3}.$$

d) Recall from part (b): taking normal vectors pointing *up*, the flux through the spherical cap S is $(8\sqrt{2} - 10)\pi/3$; the flux through the disk T is 0; the flux through the cone U is $-2\pi/3$.
The oriented boundary of the solid cone is $T - U$ (normal vectors should point out of the cone, which agrees with our previous choice for T but *not* for U). From the direct calculation in part (b), $\displaystyle\iint_{T-U} \vec{F} \cdot \hat{\mathrm{n}} \, dS = \iint_T - \iint_U = 0 + \frac{2\pi}{3} = \frac{2\pi}{3}$, which agrees with (c).
Similarly, the oriented boundary of D_2 is $S - U$. From part (b), we have
$\displaystyle\iint_{S-U} \vec{F} \cdot \hat{\mathrm{n}} \, dS = \iint_S - \iint_U = \frac{(8\sqrt{2} - 10)\pi}{3} + \frac{2\pi}{3} = \frac{(8\sqrt{2} - 8)\pi}{3}$, in agreement with (c).

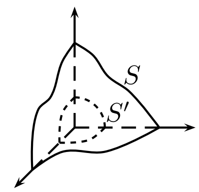
Problem 2.
a) $\vec{F} = \dfrac{-x\,\hat{\imath} - y\,\hat{\jmath} - z\,\hat{\mathrm{k}}}{(x^2 + y^2 + z^2)^{3/2}} = -\dfrac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathrm{k}}}{\rho^3}$ is directed radially inward, with length $1/\rho^2$.

b) From the geometric description, $\vec{F} \cdot \hat{\mathrm{n}} = -1/\rho^2 = -1/a^2$ on the sphere $\rho = a$. Therefore $\iint_S \vec{F} \cdot \hat{\mathrm{n}}\, dS = -\frac{1}{a^2} \iint_S dS = -\frac{1}{a^2}\, 4\pi a^2 = -4\pi$.

c) $\frac{\partial}{\partial x}(-x\rho^{-3}) = -\rho^{-3} - x \cdot (-3\,\rho_x\,\rho^{-4}) = -\rho^{-3} + 3x^2\rho^{-5}$ (using $\rho_x = x/\rho$); similarly for y and z. Therefore, $\operatorname{div} \vec{F} = -3\rho^{-3} + 3(x^2 + y^2 + z^2)\rho^{-5} = -3\rho^{-3} + 3\rho^{-3} = 0$.

The divergence theorem cannot be used to compute the flux of $\vec{F}$ over the sphere $\rho = a$, because $\vec{F}$ is not defined at every point of the interior ball ($\vec{F}$ is not defined at the origin). So there is no contradiction.

d) Consider S' = the portion of a small sphere centered at the origin which lies in the first octant, oriented outwards, and let D be the portion of the first octant between S' and the given surface S. The flux of $\vec{F}$ outwards through S' (into D) is 1/8 of that through the entire sphere (by symmetry), i.e., using the result of (b), $-4\pi/8 = -\pi/2$.



The boundary of D consists of S, $-S'$, and three flat "sides" which are portions of the coordinate planes. Because $\vec{F}$ points radially towards the origin, it is tangent to the coordinate planes, and the flux through the sides is zero. Moreover, $\operatorname{div} \vec{F} = 0$ by the result of (c), so by the divergence theorem the total flux of $\vec{F}$ out of D is zero. So:

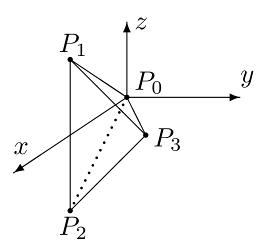
$$0 = \iint_S \vec{F} \cdot \hat{\mathrm{n}}\, dS - \iint_{S'} \vec{F} \cdot \hat{\mathrm{n}}\, dS + \iint_{\text{sides}} \vec{F} \cdot \hat{\mathrm{n}}\, dS = \iint_S \vec{F} \cdot \hat{\mathrm{n}}\, dS - (-\pi/2) + 0.$$

Hence $\iint_S \vec{F} \cdot \hat{\mathrm{n}}\, dS = -\pi/2$.

Problem 3. a) $\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathrm{k}} \\ \partial_x & \partial_y & \partial_z \\ -2xz & 0 & y^2 \end{vmatrix} = 2y\hat{\imath} - 2x\hat{\jmath}$.

b) On the unit sphere, the normal vector is $\hat{\mathrm{n}} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathrm{k}}$, so the integrand in the flux of $\operatorname{curl} \vec{F}$ is $\operatorname{curl} \vec{F} \cdot \hat{\mathrm{n}} = \langle 2y, -2x, 0 \rangle \cdot \langle x, y, z \rangle = 2xy - 2xy = 0$. Therefore, let C be a simple closed curve on the unit sphere, and let S be the portion of the surface of the sphere delimited by C. Then by Stokes' theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{\mathrm{n}}\, dS = \iint_S 0\, dS = 0$.

Problem 4.



a) For $P_0P_1P_2$: from P_0 to P_2 to P_1 back to P_0. For $P_0P_1P_3$: from P_0 to P_1 to P_3 back to P_0. For $P_0P_2P_3$: from P_0 to P_3 to P_2 back to P_0. For $P_1P_2P_3$: P_1 to P_2 to P_3 back to P_1.

b) From P_0 to P_1: $x = t$, $y = 0$, $z = t$ for $0 \le t \le 1$, so $\displaystyle\int_{P_0P_1} yz\, dy - y^2\, dz = \int_0^1 0\, dt = 0$.

From P_1 to P_3: $x = 1$, $y = t$, $z = 1 - t$ for $0 \le t \le 1$, so

$$\int_{P_1P_3} yz\, dy - y^2\, dz = \int_0^1 t(1-t)\, dt - t^2\,(-dt) = \int_0^1 t\, dt = \frac{1}{2}.$$

From P_3 to P_0: $x = t$, $y = t$, $z = 0$ for t going from 1 to 0, so $\displaystyle\int_{P_3P_0} yz\, dy - y^2\, dz = \int_1^0 0\, dt = 0$.

Therefore, the total work is $0 + \frac{1}{2} + 0 = \frac{1}{2}$.

c) $\nabla \times \vec{F} = (\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(yz))\hat{\imath} = -3y\hat{\imath}$, so by Stokes' theorem, for each face we have $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \iint_S (-3y\hat{\imath}) \cdot \hat{n}\, dS$. Hence we have to find the flux of the vector field $\vec{G} = \nabla \times \vec{F} = -3y\hat{\imath}$ through each face.

Through $P_0P_1P_2$: the face is contained in the xz-plane ($y = 0$), so the outward unit normal is $\hat{n} = -\hat{\jmath}$. Since $\vec{G} \cdot \hat{n} = 0$, the flux is zero.

Through $P_1P_2P_3$: the face is contained in the plane $x = 1$, so the outward unit normal is $\hat{n} = \hat{\imath}$, and $\vec{G} \cdot \hat{n} = \langle -3y, 0, 0\rangle \cdot \langle 1, 0, 0\rangle = -3y$.

$$\iint_{P_1P_2P_3} \vec{G}\cdot\hat{n}\, dS = \iint_{P_1P_2P_3} -3y\, dS = \int_0^1 \int_{-(1-y)}^{1-y} -3y\, dz\, dy = \int_0^1 -6y(1-y)\, dy = \Big[-3y^2 + 2y^3\Big]_0^1 = -1.$$

Through $P_0P_1P_3$: a normal vector (pointing outwards) is $\vec{N} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_3} = \langle -1, 1, 1\rangle$, so $P_0P_1P_3$ is contained in the plane $-x + y + z = 0$, i.e. the graph $z = x - y$ of $f(x, y) = x - y$; so $\hat{n}\, dS = \langle -f_x, -f_y, 1\rangle\, dx\, dy = \langle -1, 1, 1\rangle\, dx\, dy$.

So $\vec{G} \cdot \hat{n}\, dS = \langle -3y, 0, 0\rangle \cdot \langle -1, 1, 1\rangle\, dx\, dy = 3y\, dx\, dy$. The projection of the face $P_0P_1P_3$ on the xy-plane is a triangle with vertices at $(0, 0)$, $(1, 0)$ and $(1, 1)$, so

$$\iint_{P_0P_1P_3} \vec{G} \cdot \hat{n}\, dS = \int_0^1 \int_0^x 3y\, dy\, dx = \int_0^1 \frac{3}{2}x^2\, dx = \frac{1}{2}x^3\Big|_0^1 = \frac{1}{2}.$$

The symmetry $(x, y, z) \longrightarrow (x, y, -z)$ exchanges the two faces $P_0P_1P_3$ and $P_0P_2P_3$, so the two normal vectors are symmetric to each other (the orientations match). Since $\vec{G} = -3y\hat{\imath}$ is also preserved by this symmetry, the flux through $P_0P_2P_3$ is the same as through $P_0P_1P_3$, namely $1/2$.

(Or: $P_0P_2P_3$ is contained in the plane $z = -x + y$, so $\hat{n}\, dS = -\langle 1, -1, 1\rangle\, dx\, dy$ (the negative sign is so $\hat{n}$ points downwards) and $\vec{G} \cdot \hat{n}\, dS = 3y\, dx\, dy$. The projection of the face onto the xy-plane is again the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$, and the calculation proceeds as previously to give $1/2$.)

d) (i) When we add together the four answers from (c), we compute work along a curve that passes twice over each of the 6 edges of the tetrahedron. However each edge is traversed once with each orientation, so the various contributions cancel each other. For example the edge P_0P_1 is encountered once for the face $P_0P_1P_2$ (it is then oriented from P_1 to P_0) and once for the face $P_0P_1P_3$ (it is then oriented from P_0 to P_1); the sum of the two contributions is zero.

(ii) For each face $\displaystyle\oint F \cdot d\vec{r} = \iint (\operatorname{curl} \vec{F}) \cdot \hat{n}\, dS$ by Stokes, so the sum of the four line integrals is the flux of $\operatorname{curl} \vec{F} = -3y\hat{\imath}$ out of the tetrahedron. By the divergence theorem, $\displaystyle\iint_S (\operatorname{curl} \vec{F}) \cdot \hat{n}\, dS = \iiint_D \operatorname{div}(\operatorname{curl} \vec{F})\, dV$. But $\operatorname{div}(\operatorname{curl} \vec{F}) = \operatorname{div}(-3y\hat{\imath}) = 0$, so the total flux is zero.