Math 53 Homework 13

Due **Thursday 12/1/11** in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 11/21: MIDTERM 2

Wednesday 11/23: The divergence theorem (continued)

- **Read:** section 16.9.
- **Work:** 16.9: 17, 19, 27, (29).
  Problems 1 and 2 below.

Thursday 11/24 & Friday 11/25: Happy Thanksgiving!

Monday 11/28: Stokes' theorem

- **Read:** section 16.8.
- **Work:** 16.8: (1), 3, (5), (7), 9, (13), (15).
  Problems 3 and 4 below.

Wednesday 11/30: Stokes' theorem (continued)

- **Read:** sections 16.8 and 16.10.

Problem 1. Consider the space region bounded below by the right-angled cone $z = \sqrt{x^2 + y^2}$, and above by the sphere $x^2 + y^2 + z^2 = 2$. These two surfaces intersect in a horizontal circle; let T be the horizontal disk having this circle as boundary, S the spherical cap forming the upper surface, and U the cone forming the lower surface. Orient S, T, U "upwards", so the normal vector has a positive $\hat{\mathrm{k}}$-component.

a) For each of the three surfaces, determine geometrically (without calculation) whether the flux of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$ is positive or negative.

b) Calculate the flux of $\vec{F}$ across each surface (with the upwards orientation). (Do not use the divergence theorem).

c) Use the divergence theorem to find the flux of $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$ out of the solid cone bounded by T and U. Same question with the ice-cream cone bounded by S and U.

d) Show that the answers you found in part (c) are consistent with those you found in part (b). (Be careful with orientations!)

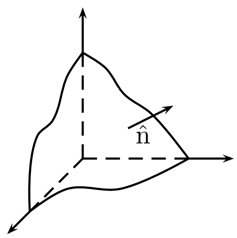
Problem 2.

a) Let $f(x, y, z) = 1/\rho = (x^2 + y^2 + z^2)^{-1/2}$. Calculate $\vec{F} = \nabla f$, and describe geometrically the vector field $\vec{F}$.

b) Evaluate the flux of $\vec{F}$ over the sphere of radius a centered at the origin.

c) Show that $\operatorname{div} \vec{F} = 0$. Does the answer obtained in (b) contradict the divergence theorem? Explain.

d) Let S be a surface in the first octant, whose boundary lies in the three coordinate planes (see picture). Show that $\iint_S \vec{F} \cdot \hat{\mathrm{n}}\, dS$ is independent of the choice of S, and calculate its value. (Hint: apply the divergence theorem to a suitable portion of the first octant).



Problem 3.

a) Calculate the curl of $\vec{F} = -2xz\hat{\imath} + y^2\hat{\mathrm{k}}$.

b) Using Stokes' theorem, show that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any simple closed curve C drawn on the unit sphere $x^2 + y^2 + z^2 = 1$.

Problem 4.

Consider the tetrahedron with vertices at $P_0 = (0, 0, 0)$, $P_1 = (1, 0, 1)$, $P_2 = (1, 0, -1)$, and $P_3 = (1, 1, 0)$.

a) Say which orientation (order of vertices) of the boundary curve of each face is compatible with the choice of the normal vector pointing out of the tetrahedron.

b) Compute the work done by the vector field $\vec{F} = yz\,\hat{\jmath} - y^2\,\hat{\mathrm{k}}$ around the boundary curve of the face $P_0P_1P_3$ directly using line integrals (using the orientation from part (a)).

c) Use Stokes' theorem to compute the work done around each of the four faces (including the one you computed directly in part (b)). Use the orientations from part (a).

(Note: the symmetry $z \to -z$ exchanges two of the faces of the tetrahedron, and can be used to avoid one calculation – if you choose to use symmetry, you need to explain why it is legitimate.)

d) The sum of the four values you found in part (c) should be zero. Explain this in two different ways:

(i) geometrically, by considering the various line integrals that are being added together;

(ii) by using the divergence theorem to compute the flux of $\operatorname{curl} \vec{F}$ out of the tetrahedron.