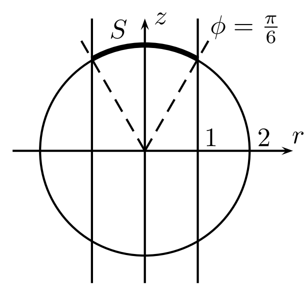
Math 53 Homework 12 – Solutions

16.7 # 14: In terms of the spherical angles ϕ and θ, we have $dS = 2^2 \sin\phi \, d\phi \, d\theta$ (cf. Example 10 in 16.6; the radius of the sphere is 2); and $y^2 = (2\sin\phi\sin\theta)^2$. The portion of sphere we are considering corresponds to $0 \le \phi \le \pi/6$ (see picture). Hence



$$\iint_S y^2 \, dS = \int_0^{2\pi}\int_0^{\pi/6} (2\sin\phi\sin\theta)^2 4\sin\phi \, d\phi \, d\theta$$
$$= 16\left(\int_0^{\pi/6} \sin^3\phi \, d\phi\right)\left(\int_0^{2\pi} \sin^2\theta \, d\theta\right).$$

$\displaystyle\int_0^{\pi/6} \sin^3\phi \, d\phi = \int_0^{\pi/6} (1-\cos^2\phi)\sin\phi \, d\phi = \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi/6} = \frac{2}{3} - \frac{3\sqrt{3}}{8}$, and
$\displaystyle\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta) \, d\theta = \pi$, so $\displaystyle\iint_S y^2 \, dS = 16\left(\frac{2}{3} - \frac{3\sqrt{3}}{8}\right)\pi = \frac{32\pi}{3} - 6\pi\sqrt{3}$.

16.7 # 17: Parametrize the cylinder by $x = x$, $y = \cos t$, $z = \sin t$: then $dS = dx \, dt$ (either by calculating $|\vec{r}_x \times \vec{r}_t| = |\langle 0, \cos t, \sin t\rangle| = 1$ or by geometry). The surface S corresponds to $0 \le x \le 3$ and $0 \le t \le \pi/2$. Then

$$\iint_S (z + x^2 y) \, dS = \int_0^3 \int_0^{\pi/2} (\sin t + x^2 \cos t) \, dt \, dx = \int_0^3 (1 + x^2) \, dx = \left[x + \frac{x^3}{3}\right]_0^3 = 12.$$

16.7 # 19: $\vec{F}(x,y,z) = \langle xy, yz, zx\rangle$, and since $z = g(x,y) = 4 - x^2 - y^2$, for the upward orientation we have $d\vec{S} = \langle -g_x, -g_y, 1\rangle \, dx \, dy = \langle 2x, 2y, 1\rangle \, dx \, dy$. Hence
$\iint_S \vec{F}\cdot d\vec{S} = \iint_S (2x^2y + 2y^2z + xz) \, dx \, dy = \int_0^1\int_0^1 (2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)) \, dy \, dx$.
Inner: $\int_0^1 -2y^4 + (8-2x^2-x)y^2 + 2x^2y + (4x - x^3) \, dy = -\frac{2}{5} + \frac{1}{3}(8 - 2x^2 - x) + x^2 + (4x - x^3)$.
Outer: $\int_0^1 (-x^3 + \frac{1}{3}x^2 + \frac{11}{3}x + \frac{34}{15}) \, dx = -\frac{1}{4} + \frac{1}{9} + \frac{11}{6} + \frac{34}{15} = \frac{713}{180}$.

16.7 # 22: $\vec{F} = \langle x, y, z^4\rangle$, and since $z = g(x,y) = \sqrt{x^2+y^2}$, with the downward orientation we have $d\vec{S} = -\langle -g_x, -g_y, 1\rangle \, dx \, dy = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\rangle \, dx \, dy$.

Moreover, the surface S lies above the unit disk $x^2 + y^2 \le 1$ in the $xy-$plane (since the cone intersects $z = 1$ where $x^2 + y^2 = 1$). Hence

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \Big(\frac{x^2+y^2}{\sqrt{x^2+y^2}} - z^4\Big) \, dx \, dy = \iint_S (r - z^4) \, r \, dr \, d\theta = \int_0^{2\pi}\int_0^1 (r - r^4) \, r \, dr \, d\theta$$

(using: $z = r$ everywhere on S). Inner: $\left[\frac{1}{3}r^3 - \frac{1}{6}r^6\right]_0^1 = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$. Outer: $2\pi \cdot \frac{1}{6} = \pi/3$.

16.7 # 23: The normal vector points radially inwards, i.e. it is negatively proportional to $\langle x, y, z\rangle$. Since $|\langle x,y,z\rangle| = \sqrt{x^2+y^2+z^2} = 2$ on S, we have $\hat{\mathrm{n}} = -\frac{1}{2}\langle x, y, z\rangle$. Hence $\vec{F}\cdot\hat{\mathrm{n}} = \langle x, -z, y\rangle \cdot (-\frac{1}{2}\langle x, y, z\rangle) = -\frac{1}{2}x^2$.
We parametrize the spherical surface using the angles ϕ and θ (first octant: $\phi \le \pi/2$, $0 \le \theta \le \pi/2$), so that $-\frac{1}{2}x^2 = -\frac{1}{2}(2\sin\phi\cos\theta)^2$ and $dS = 2^2\sin\phi \, d\phi \, d\theta$. Hence

$$\iint_S \vec{F}\cdot\hat{\mathrm{n}} \, dS = \iint_S -\frac{x^2}{2} \, dS = \int_0^{\pi/2}\int_0^{\pi/2} -8\sin^3\phi\cos^2\theta \, d\phi \, d\theta.$$

We calculate: $\int_0^{\pi/2} \sin^3\phi\,d\phi = \int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi = \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi/2} = \frac{2}{3}$, and $\int_0^{\pi/2}\sin^2\theta\,d\theta = \int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{4}$. Hence

$$\iint_S \vec{F}\cdot d\vec{S} = -8\left(\int_0^{\pi/2}\sin^3\phi\,d\phi\right)\left(\int_0^{\pi/2}\cos^2\theta\,d\theta\right) = -8(\tfrac{2}{3})(\tfrac{\pi}{4}) = -4\pi/3.$$

16.7 # 27: $\vec{F} = \langle x, 2y, 3z\rangle$. We integrate separately over each of the 6 faces, noting that they are squares of side length 2 and area 4.

- front ($x=1$): $\hat{n} = \hat{\imath}$, so $\vec{F}\cdot\hat{n} = x = 1$, and $\iint \vec{F}\cdot d\vec{S} = \iint 1\,dS = \text{area} = 4$.
- back ($x=-1$): $\hat{n} = -\hat{\imath}$, so $\vec{F}\cdot\hat{n} = -x = +1$, and $\iint \vec{F}\cdot d\vec{S} = \iint 1\,dS = \text{area} = 4$.
- right ($y=1$): $\hat{n} = \hat{\jmath}$, so $\vec{F}\cdot\hat{n} = 2y = 2$, and $\iint \vec{F}\cdot d\vec{S} = \iint 2\,dS = 2\,\text{area} = 8$.
- left ($y=-1$): $\hat{n} = -\hat{\jmath}$, so $\vec{F}\cdot\hat{n} = -2y = +2$, and $\iint \vec{F}\cdot d\vec{S} = \iint 2\,dS = 2\,\text{area} = 8$.
- top ($z=1$): $\hat{n} = \hat{k}$, so $\vec{F}\cdot\hat{n} = 3z = 3$, and $\iint \vec{F}\cdot d\vec{S} = \iint 3\,dS = 3\,\text{area} = 12$.
- bottom ($z=-1$): $\hat{n} = -\hat{k}$, so $\vec{F}\cdot\hat{n} = -3z = +3$, and $\iint \vec{F}\cdot d\vec{S} = \iint 3\,dS = 3\,\text{area} = 12$.

Summing, the total flux is $\iint_S \vec{F}\cdot d\vec{S} = 4+4+8+8+12+12 = 48$.

16.9 # 2: $\operatorname{div}\vec{F} = 2x + x + 1 = 3x+1$, so

$$\iiint_E \operatorname{div} F\,dV = \iiint_E (3x+1)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(3r\cos\theta+1)\,r\,dz\,dr\,d\theta.$$

Inner: $(3r\cos\theta+1)r(4-r^2) = 4r + 12r^2\cos\theta - r^3 - 3r^4\cos\theta$.

Middle: $\left[2r^2 + 4r^3\cos\theta - \frac{1}{4}r^4 - \frac{3}{5}r^5\cos\theta\right]_0^2 = 4 + \frac{64}{5}\cos\theta$.

Outer: $\int_0^{2\pi}(4 + \frac{64}{5}\cos\theta)\,d\theta = \left[4\theta + \frac{64}{5}\sin\theta\right]_0^{2\pi} = 8\pi$.

Next, we calculate the flux directly. Let S_1 be the surface of the paraboloid $z = 4-x^2-y^2$ ($x^2+y^2 \le 4$), oriented upwards, and S_2 the bottom face, i.e. the disk of radius 2 in the xy-plane, oriented downwards.

On S_1: the surface is $z = g(x,y) = 4-x^2-y^2$, so $d\vec{S} = \langle -g_x, -g_y, 1\rangle\,dx\,dy = \langle 2x, 2y, 1\rangle\,dx\,dy$.

$$\iint_{S_1}\vec{F}\cdot d\vec{S} = \iint_{S_1}\langle x^2, xy, z\rangle\cdot\langle 2x, 2y, 1\rangle\,dx\,dy = \iint_{S_1}(2x(x^2+y^2)+z)\,dx\,dy$$
$$= \int_0^{2\pi}\int_0^2 (2r\cos\theta(r^2) + (4-r^2))\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2(2r^4\cos\theta + 4r - r^3)\,dr\,d\theta.$$

Inner: $\left[\frac{2}{5}r^5\cos\theta + 2r^2 - \frac{1}{4}r^4\right]_0^2 = \frac{64}{5}\cos\theta + 4$. Outer: $\int_0^{2\pi}(\frac{64}{5}\cos\theta + 4)\,d\theta = 8\pi$.

On S_2: the normal vector is $\hat{n} = -\hat{k}$, and $\vec{F}\cdot\hat{n} = -z = 0$, so $\iint_{S_2}\vec{F}\cdot\hat{n}\,dS = \iint_{S_2} 0\,dS = 0$.

So the total flux is $8\pi + 0 = 8\pi$, in agreement with the first calculation.

16.9 # 4: $\operatorname{div}\vec{F} = 1+1+1 = 3$, so $\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E 3\,dV = 3(\text{volume of ball}) = 3(\frac{4}{3}\pi) = 4\pi$.

To find $\iint_S \vec{F}\cdot d\vec{S}$, we observe that the normal vector to the unit sphere is $\hat{n} = \langle x,y,z\rangle$ (pointing radially outwards), so $\vec{F}\cdot\hat{n} = \langle x,y,z\rangle\cdot\langle x,y,z\rangle = x^2+y^2+z^2 = 1$. Hence $\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S 1\,dS$. This is just the area of the unit sphere, namely 4π; or, calculating the surface integral (using $dS = \sin\phi\,d\phi\,d\theta$): $\iint_S \vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = 4\pi$.

16.9 # 7: $\operatorname{div}\vec{F} = 3y^2 + 0 + 3z^2$, so using (rotated) cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (3y^2+3z^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 3r^2\,r\,dx\,dr\,d\theta = 3(\int_0^{2\pi}d\theta)(\int_0^1 r^3\,dr)(\int_{-1}^2\,dx) = 3(2\pi)(\frac{1}{4})(3) = \frac{9\pi}{2}$.

16.9 # 11: $\operatorname{div}\vec{F} = y^2 + 0 + x^2 = x^2+y^2$, and the paraboloid and the plane intersect along the circle $z = x^2+y^2 = 4$ of radius 2, so $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3(4-r^2)\,dr\,d\theta = 2\pi\left[r^4 - \frac{1}{6}r^6\right]_0^2 = 2\pi(16 - \frac{32}{3}) = \frac{32\pi}{3}$.

Problem 1. a) North of 38° N is $0 \leq \phi < \phi_0$, where $\phi_0 = (90 - 38)\pi/180$ radians.

$$\frac{\int_0^{2\pi} \int_0^{\phi_0} \sin\phi \; d\phi d\theta}{\int_0^{2\pi} \int_0^{\pi} \sin\phi \; d\phi d\theta} = \frac{\int_0^{\phi_0} \sin\phi \; d\phi}{\int_0^{\pi} \sin\phi \; d\phi} = \frac{-\cos\phi_0 - (-1)}{2} \approx .192.$$

About 19.2 percent of the Earth's surface is north of Berkeley.

b) First take the average of ϕ:

$$\bar{\phi} = \frac{\int_0^{2\pi} \int_{\pi/2}^{\pi} \phi \sin\phi \; d\phi d\theta}{\int_0^{2\pi} \int_{\pi/2}^{\pi} \sin\phi \; d\phi d\theta} = \pi - 1$$

because $\int_{\pi/2}^{\pi} \sin\phi \; d\phi = 1$, and, by integration by parts,

$$\int_{\pi/2}^{\pi} \phi \sin\phi \, d\phi = -\int_{\pi/2}^{\pi} (-\cos\phi) \, d\phi + (-\cos\phi)\phi\Big|_{\pi/2}^{\pi} = \pi + \int_{\pi/2}^{\pi} \cos\phi \, d\phi = \pi - 1$$

Thus $\bar{\phi} = \pi - 1$ radian and the average latitude is 32.7° S. The closest large cities are Santiago, Chile (33.4° S) and Perth, Australia (31.9° S).