Math 53 Homework 10 – Solutions

16.2 # 1: $x = y^3$ and $y = t$, so $\displaystyle\int_C y^2\,ds = \int_0^2 t^2\,\sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\,dt = \int_0^2 t^3\,\sqrt{(3t^2)^2 + 1^2}\,dt$
$\displaystyle= \int_0^2 t^3\sqrt{9t^4+1}\,dt = \left[\frac{1}{54}(9t^4+1)^{3/2}\right]_0^2 = \frac{1}{54}(145^{3/2}-1).$

16.2 # 3: Parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. So
$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt = \int_{-\pi/2}^{\pi/2} 4^6\cos t\sin^4 t\,dt =$
$= 4^6\left[\frac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \frac{2}{5}4^6 = \frac{2^{13}}{5} = \frac{8192}{5}.$

16.2 # 15: On the line segment C_1 from (1,0,1) to (2,3,1): a parametric equation is $x = 1+t$, $y = 3t$, $z = 1$, for $0 \le t \le 1$. So $dx = dt$, $dy = 3dt$, $dz = 0dt$. Then
$\int_{C_1}(x+yz)\,dx + 2x\,dy + xyz\,dz = \int_0^1(1+t+3t\cdot 1)\,dt + 2(1+t)\,3\,dt + (1+t)(3t)(1)\cdot 0\,dt =$
$= \int_0^1(7+10t)\,dt = \left[7t+5t^2\right]_0^1 = 12.$
On the line segment C_2 from (2,3,1) to (2,5,2): $x = 2$, $y = 3+2t$, $z = 1+t$ for $0 \le t \le 1$. So $dx = 0\,dt$, $dy = 2\,dt$, $dz = dt$, and
$\int_{C_2}(x+yz)\,dx + 2x\,dy + xyz\,dz = \int_0^1(2+(3+2t)(1+t))\cdot 0\,dt + (2\cdot 2)\cdot 2\,dt + 2(3+2t)(1+t)\,dt$
$= \int_0^1(14+10t+4t^2)\,dt = \left[14t+5t^2+\frac{4}{3}t^3\right]_0^1 = \frac{61}{3}.$
So $\int_C(x+yz)\,dx + 2x\,dy + xyz\,dz = \int_{C_1} + \int_{C_2} = 12 + \frac{61}{3} = \frac{97}{3}.$

16.2 # 17: (a) Along the line $x = -3$, the vectors of $\vec{F}$ have positive y-components; since the path goes upward, the integrand $\vec{F}\cdot\hat{T}$ is always positive. So $\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot\hat{T}\,ds > 0$.
(b) Along the circle C_2, the field vectors are all pointing in the clockwise direction, i.e., opposite the direction of the path. So $\vec{F}\cdot\hat{T} < 0$, and therefore $\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{C_2}\vec{F}\cdot\hat{T}\,ds < 0$.

16.2 # 22: $x = t$, $y = \sin t$, $z = \cos t$, so $dx = dt$, $dy = \cos t\,dt$, $dz = -\sin t\,dt$. Hence
$\int_C\vec{F}\cdot d\vec{r} = \int_C z\,dx + y\,dy - x\,dz = \int_0^\pi \cos t\,dt + \sin t\cos t\,dt - t(-\sin t)\,dt.$
Integration by parts yields: $\int t\sin t\,dt = -t\cos t - \int 1(-\cos t)\,dt = -t\cos t + \sin t$.
So $\int_C\vec{F}\cdot d\vec{r} = \int_0^\pi(\cos t + \sin t\cos t + t\sin t)\,dt = \left[\sin t + \frac{1}{2}\sin^2 t - t\cos t + \sin t\right]_0^\pi = \pi.$

16.2 # 32: (a) We parametrize the circle C by: $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. So $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$, and
$\int_C\vec{F}\cdot d\vec{r} = \int_C x^2\,dx + xy\,dy = \int_0^{2\pi}(2\cos t)^2\,(-2\sin t)\,dt + (4\cos t\sin t)\,(2\cos t)\,dt$
$= \int_0^{2\pi}(-8\cos^2 t\sin t + 8\cos^2 t\sin t)\,dt = \int_0^{2\pi} 0\,dt = 0.$
(b) The vector $\vec{F}(x,y) = x^2\hat{\imath} + xy\hat{\jmath} = x(x\hat{\imath} + y\hat{\jmath})$ is parallel to the position vector $x\hat{\imath} + y\hat{\jmath}$ of the point (x,y), so it always points in the radial direction (straight away from the origin if $x > 0$, straight towards the origin if $x < 0$). (This can be seen on a plot).
So, at every point of the circle C, the vector $\vec{F}(x,y)$ is perpendicular to the circle, hence the field does no work on the moving particle. In other words, $\vec{F}\cdot\hat{T} = 0$ at any point along C, and so $\int_C\vec{F}\cdot d\vec{r} = 0$.

16.2 # 41: The line segment from (1,0,0) to (3,4,2) has parametric equations $x = 1+2t$, $y = 4t$, $z = 2t$ for $0 \le t \le 1$; so $dx = 2dt$, $dy = 4dt$, $dz = 2dt$, and
$\int_C\vec{F}\cdot d\vec{r} = \int_C(y+z)\,dx + (x+z)\,dy + (x+y)\,dz = \int_0^1(6t)\,2dt + (1+4t)\,4dt + (1+6t)\,2dt =$
$\int_0^1(40t+6)\,dt = \left[20t^2+6t\right]_0^1 = 26.$

16.3 # 3: $\frac{\partial}{\partial y}(2x-3y) = -3 = \frac{\partial}{\partial x}(-3x+4y-8)$, and $\vec{F}$ is defined in the entire plane (open and simply connected), so $\vec{F}$ is conservative. So there is a function f such that $\nabla f = \vec{F}$, i.e. $f_x = 2x - 3y$ and $f_y = -3x + 4y - 8$.

Integrating with respect to x, $f_x(x,y) = 2x - 3y$ implies $f(x,y) = x^2 - 3xy + g(y)$ for some function $g(y)$; differentiating both sides of this equation with respect to y gives $f_y = -3x + g'(y)$. Thus we should have $-3x + g'(y) = -3x + 4y - 8$, or $g'(y) = 4y - 8$. Therefore $g(y) = 2y^2 - 8y + c$ where c is a constant. Hence $f(x,y) = x^2 - 3xy + 2y^2 - 8y + c$.

16.3 # 8: $\frac{\partial}{\partial y}(xy\cos xy + \sin xy) = x\cos xy - x^2 y \sin xy + x\cos xy = 2x\cos xy - x^2 y\sin xy$, while $\frac{\partial}{\partial x}(x^2\cos xy) = 2x\cos xy - x^2 y \sin xy$. So $\vec{F}$ is defined everywhere and satisfies $P_y = Q_x$, and hence it is conservative.

We now look for f such that $\nabla f = \vec{F}$, i.e. $f_x = xy\cos xy + \sin xy$ and $f_y = x^2\cos xy$. Integrating with respect to y, $f_y = x^2 \cos xy$ implies that $f(x,y) = x\sin xy + g(x)$ for some function $g(x)$. Differentiating this equation with respect to x, we get $f_x = \sin xy + x^2\cos xy + g'(x)$; since we want $f_x = xy\cos xy + \sin xy$, we deduce that $g'(x) = 0$, i.e. $g(x) = c$ is a constant. Thus $f(x,y) = x\sin xy + c$.

(The usual method, namely first integrating $f_x = P$ with respect to x to find f up to a function of y, and then differentiating with respect to y to find that function, would also work just fine. However in this example it is slightly easier to integrate Q with respect to y than to integrate P with respect to x.)

16.3 # 15: a) $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$, and so $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$, so $g_y(y,z) = 0$, and (integrating with respect to y) $g(y,z) = h(z)$. Thus $f(x,y,z) = xyz + h(z)$, and $f_z(x,y,z) = xy + h'(z)$. But $f_z(x,y,z) = xy + 2z$, so $h'(z) = 2z$, and hence $h(z) = z^2 + c$. Thus (taking $c = 0$), one potential function is $f(x,y,z) = xyz + z^2$.

b) By the fundamental theorem, $\int_C \vec{F}\cdot d\vec{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$.

16.3 # 19: Note that $\vec{F} = \langle \tan y, x\sec^2 y\rangle$ is only defined when $y \neq \frac{\pi}{2} + n\pi$. Therefore the path C from $(1,0)$ to $(2,\pi/4)$ can't be arbitrary, it must lie in the region $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

With this understood, we note that $\frac{\partial}{\partial y}(\tan y) = \sec^2 y = \frac{\partial}{\partial x}(x\sec^2 y)$, and the region $-\frac{\pi}{2} < y < \frac{\pi}{2}$ is simply connected, so $\vec{F}$ is conservative.

Using the usual method, we find that $f(x,y) = x\tan y$ is a potential function fo $\vec{F}$.

Therefore $\int_C \tan y\,dx + x\sec^2 y\,dy = \int_C \vec{F}\cdot d\vec{r} = f(2,\pi/4) - f(1,0) = 2 - 0 = 2$.

16.3 # 21: $\frac{\partial}{\partial y}(2y^{3/2}) = 3y^{1/2} = \frac{\partial}{\partial x}(3x\sqrt{y})$, and the region $y > 0$ where $\vec{F}$ is continuously differentiable is simply connected. Thus $\vec{F}$ is conservative and can be written as ∇f for some potential function $f(x,y)$. The equation $f_x = 2y^{3/2}$ implies that $f(x,y) = 2xy^{3/2} + g(y)$. Hence $f_y = 3xy^{1/2} + g'(y) = 3xy^{1/2}$, and so $g'(y) = 0$, i.e. g is constant. Thus $f(x,y) = 2xy^{3/2}$ is a potential function. Therefore, given any path C from (1,1) to (2,4) (in the upper half plane $y > 0$), $\int_C \vec{F}\cdot d\vec{r} = f(2,4) - f(1,1) = 32 - 2 = 30$.

16.3 # 23: We know that if the vector field (call it $\vec{F}$) is conservative, then around any closed path C we must have $\int_C \vec{F}\cdot d\vec{r} = 0$. However, take C to be a circle centered at the origin, oriented counterclockwise. Then all of the field vectors along C point forward (have a positive tangential component), so $\int_C \vec{F}\cdot d\vec{r} > 0$. Therefore the field is not conservative.

16.4 # 2: a) Let C_1 = bottom edge of the rectangle from (0,0) to (3,0), C_2 = right edge from (3,0) to (3,1), C_3 = top from (3,1) to (0,1), C_4 = left edge from (0,1) back to (0,0).
C_1: $x = t$, $y = 0$, for $0 \le t \le 3$, so $dx = dt$, $dy = 0$, and $\int_{C_1} xy\,dx + x^2\,dy = \int_0^3 0\,dt = 0$.
C_2: $x = 3$, $y = t$, for $0 \le t \le 1$, so $dx = 0$, $dy = dt$, and $\int_{C_2} xy\,dx + x^2\,dy = \int_0^1 9\,dt = 9$.
C_3: $x = 3 - t$, $y = 1$, $0 \le t \le 3$; so $dx = -dt$, $dy = 0$, and

$$\int_{C_3} xy\,dx + x^2\,dy = \int_0^3 -(3-t)\,dt = \left[-3t + \tfrac{1}{2}t^2\right]_0^3 = -9/2.$$

C_4: the integrand vanishes because $x = 0$, so $\int_{C_4} xy\,dx + x^2\,dy = 0$.
Adding these together, $\oint_C xy\,dx + x^2\,dy = 0 + 9 - \frac{9}{2} + 0 = 9/2$.
b) by Green's theorem, $\displaystyle\oint_C xy\,dx + x^2\,dy = \iint_R \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right) dA = \iint_R x\,dA$, where R is the rectangle $0 \le x \le 3$, $0 \le y \le 1$. So $\iint_R x\,dA = \int_0^3 \int_0^1 x\,dy\,dx = \int_0^3 x\,dx = 9/2$.

16.4 # 4: a) Let C_1 be the segment from $(0,1)$ to $(0,0)$, C_2 the segment from $(0,0)$ to $(1,0)$, and C_3 the parabola $y = 1 - x^2$ from $(1,0)$ to $(0,1)$.
C_1: $x = 0$, $y = 1 - t$ for $0 \le t \le 1$, so $dx = 0$ and $dy = -dt$; hence $\int_{C_1} x\,dx + y\,dy = \int_0^1 -(1-t)\,dt = \left[-t + \frac{1}{2}t^2\right]_0^1 = -\frac{1}{2}$.
C_2: $x = t$, $y = 0$ for $0 \le t \le 1$, so $dx = dt$ and $dy = 0$. Hence $\int_{C_2} x\,dx + y\,dy = \int_0^1 t\,dt = \frac{1}{2}$.
C_3: $x = 1 - t$, $y = 1 - (1-t)^2 = 2t - t^2$, for $0 \le t \le 1$; so $dx = -dt$ and $dy = (2 - 2t)\,dt$. Hence $\int_{C_3} x\,dx + y\,dy = \int_0^1 -(1-t)\,dt + (2t - t^2)(2 - 2t)\,dt = \int_0^1 (2t^3 - 6t^2 + 5t - 1)\,dt = \left[\frac{1}{2}t^4 - 2t^3 + \frac{5}{2}t^2 - t\right]_0^1 = 0$.
So $\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} = -\frac{1}{2} + \frac{1}{2} + 0 = 0$.
(Note: switching the orientations of C_1 and C_3 would have given slightly simpler parametrizations; however one then needs to be careful about signs when adding up the three portions.)
b) $\displaystyle\oint_C x\,dx + y\,dy = \iint_R \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right) dA = \iint_R 0\,dA = 0$.

16.4 # 9: $\displaystyle\int_C y^3\,dx - x^3\,dy = \iint_R \left(\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right) dA = \iint_R (-3x^2 - 3y^2)\,dA$, where R is the disk $x^2 + y^2 \le 4$.
So $\iint_R (-3x^2 - 3y^2)\,dA = \int_0^{2\pi} \int_0^2 -3r^2\,r\,dr\,d\theta = 2\pi \cdot \left[-\frac{3}{4}r^4\right]_0^2 = -24\pi$.

16.4 # 12: C is clockwise; its edges are parts of the lines $y = 3x$, $x = 2$, and $y = 0$ respectively, so C encloses the region R defined by the inequalities $0 \le y \le 3x$, $0 \le x \le 2$.
Therefore $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = -\oint_{-C} \vec{F} \cdot d\vec{r} = -\iint_R \left(\frac{\partial}{\partial x}(x^2 + 2y\sin x) - \frac{\partial}{\partial y}(y^2 \cos x)\right) dA =$

$$= -\iint_R 2x\,dA = -\int_0^2 \int_0^{3x} 2x\,dy\,dx = -\int_0^2 6x^2\,dx = -\left[2x^3\right]_0^2 = -16.$$

16.4 # 19: Let C_1 be the arch of cycloid from $(0,0)$ to $(2\pi, 0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi, 0)$ to $(0,0)$ (so C_2 is given by $x = 2\pi - t$, $y = 0$ for $0 \le t \le 2\pi$). Then $C = C_1 + C_2$ is traversed clockwise, so $-C$ is oriented positively and encloses the area under one arch of the cycloid. By formula (5) on p. 1058,
$A = \oint_{-C} -y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt + \int_0^{2\pi} 0\,(-dt) =$
$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt = \int_0^{2\pi} (1 - 2\cos t + \frac{1+\cos t}{2})\,dt = \left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi$.

16.4 # 25: Let D be the region enclosed by C. Then by Green's theorem,

$$-\frac{\rho}{3}\oint_C y^3\,dx = -\frac{\rho}{3}\iint_D -\frac{\partial}{\partial y}(y^3)\,dA = \frac{\rho}{3}\iint_D 3y^2\,dA = \iint_A y^2\,\rho\,dA = I_x.$$

Similarly, $\displaystyle \frac{\rho}{3}\oint_C x^3\,dy = \frac{\rho}{3}\iint_D \frac{\partial}{\partial x}(x^3)\,dA = \frac{\rho}{3}\iint_D 3x^2\,dA = \iint_A x^2\,\rho\,dA = I_y.$

Problem 1: $\displaystyle \int_C \vec{F}\cdot d\vec{r} = \int_C (x^2y + \frac{1}{3}y^3)\,dx = \int_{x_1}^{x_2}\left(x^2 f(x) + \frac{1}{3}f(x)^3\right)dx,$ and

$$\iint_R (x^2+y^2)\,dA = \int_{x_1}^{x_2}\int_0^{f(x)}(x^2+y^2)\,dy\,dx = \int_{x_1}^{x_2}\left[x^2y + \frac{1}{3}y^3\right]_0^{f(x)}dx$$
$$= \int_{x_1}^{x_2}\left(x^2 f(x) + \frac{1}{3}f(x)^3\right)dx.$$

These two integrals are therefore equal.

Problem 2. a) For $\theta(x,y) = \tan^{-1}(y/x)$:

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\,\frac{-y}{x^2} = -\frac{y}{x^2+y^2}, \text{ and } \frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2}; \text{ so } \nabla\theta = \vec{F}.$$

b) Because $\theta(x,y) = \tan^{-1}(y/x)$ is well-defined in the right half-plane ($x>0$) and $\vec{F} = \nabla\theta$, the fundamental theorem for line integrals implies $\int_C \vec{F}\cdot d\vec{r} = \theta(x_2,y_2) - \theta(x_1,y_1) = \theta_2 - \theta_1$.

c) $\displaystyle \int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_1}\frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{\pi}\frac{(-\sin\theta)(-\sin\theta)+\cos\theta\cos\theta}{\cos^2\theta+\sin^2\theta}\,d\theta = \int_0^{\pi}d\theta = \pi.$

Similarly, $\displaystyle \int_{C_2}\vec{F}\cdot d\vec{r} = \int_0^{-\pi}d\theta = -\int_{-\pi}^{0}d\theta = -\pi.$

(Or geometrically: length(C_1) = length(C_2) = π, $\vec{F}\cdot\hat{T} = 1$ on C_1; $\vec{F}\cdot\hat{T} = -1$ on C_2)

d) $\vec{F}$ is defined everywhere except at the origin, but is not conservative over its entire domain of definition. Indeed, the two line integrals computed in (c) both run from (1,0) to $(-1,0)$ but they are not equal, so path-independence fails. On the other hand, $\vec{F}$ is conservative over the half-plane $x>0$, where $\vec{F} = \nabla\theta$ and the fundamental theorem of calculus gives a formula for the line integral involving only the values of θ at the end points (as seen in (b)).

e) $\displaystyle \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, while

$\displaystyle \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}$. So $P_y = Q_x$.

f) By Green's theorem, if C is a simple closed curve enclosing a region R of the xy-plane which does not contain the origin, then $\displaystyle \int_C \vec{F}\cdot d\vec{r} = \iint_R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \iint_R 0\,dA = 0.$

The argument does not apply when R contains the origin: in that case $\vec{F}$ is not continuously differentiable everywhere in R, and Green's theorem does not apply. For instance, if C is the unit circle oriented counterclockwise, then using (c), $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} - \int_{C_2}\vec{F}\cdot d\vec{r} = \pi - (-\pi) = 2\pi \neq 0$. (This also holds for more general curves, see Example 5 on p. 1059).

Note: the fact that $P_y = Q_x$ does not imply that $\vec{F}$ is the gradient of a well-defined potential function everywhere! This would only be true if $\vec{F}$ were defined over the entire plane (or more generally, a *simply connected region*). In fact, we can find a potential function for $\vec{F}$ over smaller regions such as the right half-plane $x>0$ (namely, the polar angle θ). However, if we consider the entire plane with just the origin removed, the polar angle coordinate θ is not well-defined as a single-valued differentiable function: its value "jumps" by 2π as we go around the origin. This is what causes conservativeness to fail.

Problem 3: a) If $\vec{F} = r^n(x\,\hat{\imath} + y\,\hat{\jmath}) = P\hat{\imath} + Q\hat{\jmath}$ then

$$Q_x = \frac{\partial(yr^n)}{\partial x} = nyr^{n-1}\frac{x}{r}, \text{ while } P_y = \frac{\partial(xr^n)}{\partial y} = nxr^{n-1}\frac{y}{r}. \quad \text{So } P_y = Q_x.$$

(Recall $r = \sqrt{x^2+y^2}$ gives $r_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}$, and similarly $r_y = \dfrac{y}{r}$.)

b) If $g = g(r)$, then $g_x = g'(r)\,\dfrac{x}{r}$ and $g_y = g'(r)\,\dfrac{y}{r}$ (by the chain rule).

So $\nabla g = \dfrac{g'(r)}{r}(x\,\hat{\imath} + y\,\hat{\jmath})$. We must find g such that $g'(r)/r = r^n$, i.e. $g'(r) = r^{n+1}$.

Two cases: $n \neq -2$: $g(r) = \dfrac{1}{n+2}\,r^{n+2}$. $\qquad n = -2$: $g(r) = \ln(r)$.