Math 53 Homework 10

Due Tuesday 11/8/11 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 10/31: Line integrals

- **Read:** section 16.2.
- **Work:** 16.2: 1, 3, (7), (11), 15, 17, 22, (29), 32*, (39), 41, (45).
  Problem 1 below.

* For 16.2 # 32: for part (b), try to find a geometric argument instead! What is the direction of $\vec{F}$? Observe: $\vec{F} = x(x\hat{\imath} + y\hat{\jmath})$.

Wednesday 11/2: Gradient fields, fundamental theorem for line integrals

- **Read:** section 16.3.
- **Work:** 16.3: 3, (5), (7), 8, (11), (13), 15, (17), 19, 21, 23, (27).
  Problems 2 and 3 below.

Friday 11/4: Green's theorem

- **Read:** section 16.4.
- **Work:** 16.4: (1), 2, (3), 4, (7), 9, 12, (13), (17), 19, (21), 25, (26).

Problem 1.

Consider the vector field $\vec{F} = (x^2y + \frac{1}{3}y^3)\,\hat{\imath}$, and let C be the portion of the graph $y = f(x)$ running from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ (assume that $x_1 < x_2$, and f takes positive values). Show that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is equal to the polar moment of inertia of the region R lying below C and above the x-axis (with density $\rho = 1$).

Problem 2. Consider the vector field $\vec{F}(x, y) = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2 + y^2}$.

a) Show that $\vec{F}$ is the gradient of the polar angle function $\theta(x, y) = \tan^{-1}(y/x)$ defined over the right half-plane $x > 0$. (Note: this formula for θ does not make sense for $x = 0$!)

b) Suppose that C is a smooth curve in the right half-plane $x > 0$ joining two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$. Express $\int_C \vec{F} \cdot d\vec{r}$ in terms of the polar coordinates (r_1, θ_1) and (r_2, θ_2) of A and B.

c) Compute directly from the definition the line integrals $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_2} \vec{F} \cdot d\vec{r}$, where C_1 is the upper half of the unit circle running from $(1, 0)$ to $(-1, 0)$, and C_2 is the lower half of the unit circle, also going from $(1, 0)$ to $(-1, 0)$.

d) Using the results of parts (a)-(c), is $\vec{F}$ conservative (path-independent) over its entire domain of definition? Is it conservative over the right half-plane $x > 0$? Justify your answers.

e) Show that the components P and Q of $\vec{F}$ satisfy the equation $\partial P/\partial y = \partial Q/\partial x$ at any point of the plane where $\vec{F}$ is defined (not just in the right half-plane $x > 0$).

f) (After Friday's lecture) Show that $\int_C \vec{F} \cdot d\vec{r} = 0$ for every simple closed curve that does not pass through or enclose the origin. Does this remain true if C encloses the origin?

Note: in fact it is true that $\vec{F} = \nabla\theta$ everywhere. However, the polar angle θ cannot be defined as a single-valued differentiable function everywhere (if you try, you will find that it is only well-defined up to adding multiples of 2π). This is why in parts (a) and (b) we only consider the right half-plane; any other region over which θ can be defined unambiguously in a continuous manner would be equally suitable.

Problem 3.

a) For which values of n do the components P and Q of $\vec{F} = r^n(x\,\hat{\imath} + y\,\hat{\jmath})$ satisfy $\partial P/\partial y = \partial Q/\partial x$? (Here $r = \sqrt{x^2 + y^2}$; start by finding formulas for r_x and r_y).

b) Whenever possible, find a function g such that $\vec{F} = \nabla g$. (Hint: look for a function of the form $g = g(r)$, with $r = \sqrt{x^2 + y^2}$. Watch out for a certain negative value of n for which the general formula doesn't work.)