MIRROR SYMMETRY: LECTURE 8

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Last time: 18.06 Linear Algebra.
Today: 18.02 Multivariable Calculus. / 18.04 Complex Variables
Thursday: 18.03 Differential Equations

1. Mirror Symmetry Conjecture

Last time, we said that if we have a large complex structure limit (LCSL) degeneration, then we have a special basis $(\alpha_0, \dots, \alpha_S, \beta_0, \dots, \beta_S)$ of $H_3(X, \mathbb{Z})$ s.t. β_0 is invariant under monodromy and $\beta_1, \dots, \beta_s$ are mapped by monodromy by $\beta_i \xrightarrow{\phi_j} \beta_i - m_{ji}\beta_0$ for $m_{ji} \in \mathbb{Z}$. We decided that we would normalize so that $\int_{\beta_0} \Omega = 1$, and let $w_i = \int_{\beta_i} \Omega$ ($w_i \xrightarrow{\phi_j} w_i - m_{ji}$) and $q_i = \exp(2\pi i w_i)$ (which we called canonical coordinates).

Example. Given a family of tori T^2 with monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\int_a \Omega = 1, \int_b \Omega = \tau$ (precisely what you get identifying the elliptic curve with $\mathbb{R}^2/\mathbb{Z} \oplus \tau\mathbb{Z}$), $q = \exp(2\pi i \tau)$.

If e_i is a basis of $H^2(\check{X}, \mathbb{Z})$, e_i in the Kähler cone, we obtain coordinates on the complexified Kähler moduli space: if $[B + i\omega] = \sum \check{t}_i e_i$, let $\check{q}_i = \exp(2\pi i \check{t}_i), \check{t}_i = \int_{e_i^*} B + i\omega$.

Example. Returning to our example, $\check{q} = \exp(2\pi i \int_{T^2} B + i\omega)$.

Conjecture 1 (Mirror Symmetry)**.** *Let $f : \mathcal{X} \to (D^*)^S$ be a family of Calabi-Yau 3-folds with LCSL at 0. Then $\exists$ a Calabi-Yau 3-fold $\check{X}$ and choices of bases $\alpha_0, \dots, \alpha_S, \beta_0, \dots, \beta_S$ of $H_3(X, \mathbb{Z})$, $e_1, \dots, e_S$ of $H^2(X, \mathbb{Z})$ s.t. under the map $m : (D^*)^S \to \mathcal{M}_{Kah}(\check{X})$, $(q_1, \dots, q_S) \mapsto (\check{q}_i, \dots, \check{q}_S)$, $\check{q}_i = q_i$, we have a coincidence of Yukawa couplings*

$$\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \rangle_p^X = \langle \frac{\partial}{\partial \check{q}_i}, \frac{\partial}{\partial \check{q}_j}, \frac{\partial}{\partial \check{q}_k} \rangle_{m(p)}^{\check{X}} \tag{1}$$

where the LHS corresponds to $\int_X \Omega \wedge (\frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} \Omega)$ and the RHS to a $(1,1)$-coupling, i.e. the Gromov-Witten invariants $\langle e_i, e_j, e_k \rangle_{0,\beta}^{\check{X}}$ (since $2\pi i \check{q}_i \frac{\partial}{\partial \check{q}_i} = \frac{\partial}{\partial \check{t}_i} = e_i \in H^{1,1}$ etc.).

Remark. A more grown-up version of mirror symmetry would give you an equivalence between $H^*(X, \bigwedge TX)$ with its usual product structure and $H^*(\check{X}, \mathbb{C})$ with the quantum twisted product structure as Frobenius algebras (making this concrete would require more work).

1.1. **Application to the Quintic (See Gross-Huybrechts-Joyce, after Candelas-de la Ossa-Greene-Parkes).** Last time, we defined

$$X_\psi = \{(x_0 : \cdots : x_4) \in \mathbb{P}^4 \mid f_\psi = \sum_0^4 x_i^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0\} \tag{2}$$

with

$$G = \{(a_0, \ldots, a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 \mid \sum a_i = 0\}/\{(a,a,a,a,a)\} \cong (\mathbb{Z}/5\mathbb{Z})^3 \tag{3}$$

acting by diagonal multiplication $x_i \mapsto x_i \xi^{a_i}, \xi = e^{2\pi i/5}$. We obtained a crepant resolution $\check{X}_\psi$ of X_ψ/G (its singularities are $\overline{C_{ij}} = \{x_i = x_j = 0\}/G$), which has $h^{1,1} = 101$, $h^{2,1} = 1$, and $h^3 = 4$. The map $(x_0 : \ldots : x_4) \mapsto (\xi^a x_0 : x_1 : \ldots : x_4)$ gives $X_\psi \cong X_{\xi\phi}$, so let $z = (5\xi)^{-5}$. Then $z \to 0$, i.e. $\psi \to \infty$, gives a toric degeneration of X_ψ to $\{x_0 x_1 x_2 x_3 x_4 = 0\}$. This is maximally unipotent, as the monodromy on H^3 is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

so it is LCSL. We want to compute the *periods* of the holomorphic volume form on $\check{X}_\psi$. There is a volume form $\check{\Omega}_\psi$ on $\check{X}_\psi$ induced by the G-invariant volume form Ω_ψ on X_ψ by pullback via $\pi : \check{X}_\psi \to X_\psi/G$. We want to find a 3-cycle $\beta_0 \in H_3(\check{X}_\psi)$ that survives the degeneration. For $z = 0$, $\{\prod x_i = 0\}$ contains tori in component $\mathbb{P}^3$'s, e.g.

$$T_0 = \{(x_0 : \cdots : x_4) \mid x_4 = 1, |x_0| = |x_1| = |x_2| = \delta, x_3 = 0\} \tag{5}$$

We want to extend it to $z \neq 0$. Take $x_4 = 1, |x_0| = |x_1| = |x_2| = \delta$: then x_3 should be given by the root of f_ψ which tends to 0 as $\psi \to \infty$. We need to show that there is only one such value (giving us a simple degeneration rather than a branched covering). Explicitly, set $x_3 = (\psi x_0 x_1 x_2)^{1/4} y$:

$$f_\psi = 0 \Leftrightarrow x_0^5 + x_1^5 + x_2^5 + (\psi x_0 x_1 x_2)^{5/4} y^5 + 1 - 5(\psi x_0 x_1 x_2)^{5/4} y \tag{6}$$

i.e.

$$y = \frac{y^5}{5} + \frac{x_0^5 + x_1^5 + x_2^5 + 1}{5(\psi x_0 x_1 x_2)^{5/4}} \tag{7}$$

One root is $y \sim \psi^{-5/4} \to 0$, with the other four roots converging to $\sqrt[4]{5}$. So for x_3, we have one root $\sim \psi^{-1}$, and 4 roots $\sim \psi^{1/4}$. Now, G acts freely on $T_0 \subset X_\psi$, and T_0/G is contained in the smooth part of X_ψ/G and gives a torus $\check{T}_0 \subset \check{X}_\psi, \beta_0 = [\check{T}_0]$. Because $T_0, \check{T}_0$ still make sense at $z = 0$, their class is preserved by the monodromy.

Next, we want to get the required holomorphic volume form. In the affine subset $x_4 = 1$, let Ω_ψ be the 3-form on X_ψ characterized uniquely by

$$\Omega_\psi \wedge df_\psi = 5\psi dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \tag{8}$$

at each point of X_ψ. At a point where $\frac{\partial f_\psi}{\partial x_3} \neq 0$, (x_0, x_1, x_2) are local coordinates, and

$$\Omega_\psi = \frac{5\psi dx_0 \wedge dx_1 \wedge dx_2}{\frac{\partial f_\psi}{\partial x_3}} = \frac{5\psi dx_0 \wedge dx_1 \wedge dx_2}{5x_3^4 - 5\psi x_0 x_1 x_2} \tag{9}$$

Defining it in terms of other coordinates, we get the same formula on restrictions. We need to extend this to where $x_4 = 0$. We could rewrite this using homogeneous coordinates, but note that the corresponding divisor is just the canonical divisor: since X_ψ is Calabi-Yau, this divisor has no zeroes or poles at $x_4 = 0$. Since Ω_ψ is G-invariant, it induces a 3-form on $(X_\psi/G)^{\text{nonsing}}$ and lifts and extends to $\check{\Omega}_\psi$ on $\check{X}_\psi$ with

$$\int_{\check{T}_0} \check{\Omega}_\psi = \frac{1}{5^3} \int_{T_0} \Omega_\psi \tag{10}$$

Tool: we have the residue formula

$$\frac{1}{2\pi i} \int_{S^1} f(z)dz = \sum_{z_i \text{ poles of } f \in D^2} \text{res}_f(z_i) \tag{11}$$

So let $T^4 = \{|x_0| = |x_1| = |x_2| = |x_3| = \delta, x_4 = 1\}$. Then

$$\frac{1}{2\pi i} \int_{T^4} \frac{5\psi dx_0 dx_1 dx_2 dx_3}{f_\psi} = \int_{T^3_{x_0x_1x_2}} \left(\frac{1}{2\pi i} \int_{S^1} \frac{5\psi dx_3}{f_\psi} \right) dx_0 dx_1 dx_2 \tag{12}$$

where f_ψ has a unique pole at x_3. The residue is precisely $\frac{5\psi}{(\partial f/\partial x_3)}$, giving us

$$= \int_{T_0} \frac{5\psi}{(\partial f/\partial x_3)} dx_0 dx_1 dx_2 = \int_{T_0} \Omega_\psi \tag{13}$$

So

$$\begin{aligned}\int_{T_0}\Omega_\psi &= \frac{1}{2\pi i}\int_{T^4}\frac{dx_0dx_1dx_2dx_3}{(5\psi)^{-1}(x_0^5+x_1^5+x_2^5+x_3^5+1)-x_0x_1x_2x_3}\\ &= -\frac{1}{2\pi i}\int_{T^4}\frac{dx_0dx_1dx_2dx_3}{x_0x_1x_2x_3}\left(1-(5\psi)^{-1}\frac{x_0^5+x_1^5+x_2^5+x_3^5+1}{x_0x_1x_2x_3}\right)^{-1}\\ &= -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{T^4}\frac{dx_0dx_1dx_2dx_3}{x_0x_1x_2x_3}\cdot\frac{(x_0^5+x_1^5+x_2^5+x_3^5+1)^m}{(5\psi)^m(x_0x_1x_2x_3)^m}\end{aligned} \tag{14}$$

We want to find the coefficient of 1 in the latter term. We obviously need $m = 5n$ (the numerator only has powers which are a multiple of 5), and want the coefficient of $x_0^{5n}x_1^{5n}x_2^{5n}x_3^{5n}$ in $(x_0^5+x_1^5+x_2^5+x_3^5+1)^{5n}$, which is $\frac{(5n)!}{(n!)^5}$. We finally obtain

$$\int_{T_0}\Omega_\psi = -(2\pi i)^3\sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5(5\psi)^{5n}} \tag{15}$$

In terms of $z = (5\psi)^{-5}$, the period is proportional to

$$\phi_0(z) = \sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}z^n \tag{16}$$

Set $a_n = \frac{(5n)!}{(n!)^5}$. Then

$$(n+1)^4 a_{n+1} = \frac{(5n+5)!}{(n!)^5(n+1)} = 5(5n+4)(5n+3)(5n+2)(5n+1)a_n \tag{17}$$

Setting $\Theta = z\frac{d}{dz} : \Theta(\sum c_n z^n) = \sum n c_n z^n$, giving us the *Picard-Fuchs equation*

$$\Theta^4\phi_0 = 5z(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)\phi_0 \tag{18}$$

Next time, we will show that there is a unique regular solution, and a unique solution with logarithmic poles to our original problem.