MIRROR SYMMETRY: LECTURE 23

DENIS AUROUX
NOTES BY KARTIK VENKATRAM

Recall that given an (almost) Calabi-Yau manifold (X, J, ω, Ω), we defined M to be the set of pairs (L, ∇), $L \subset X$ a special Lagrangian torus, ∇ a flat $U(1)$ connection on $\mathbb{C} \times L$ modulo gauge equivalence. Up to 2π,

$$\begin{aligned} T_{(L,\nabla)}M &= \{(v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L; i\mathbb{R}) \mid -\iota_v\omega + \frac{1}{2\pi} i\alpha \in \mathcal{H}^1_\psi(L; \mathbb{C})\} \\ &= H^1_\psi(L, \mathbb{C}) = \{\beta \in \Omega^1(L, \mathbb{R}) \mid d\beta = 0, d^*(\psi\beta) = 0\}, \psi = |\Omega|_g \end{aligned} \tag{1}$$

is a complex vector space, giving us an integrable $J^\vee$ on M with holomorphic local coordinates

$$z_\beta(L, \nabla) = \underbrace{\exp(-2\pi\omega(\beta))}_{\mathbb{R}_+} \underbrace{\mathrm{hol}_\nabla(\gamma)}_{U(1)} \in \mathbb{C}^* \tag{2}$$

and a holomorphic $(n, 0)$-form

$$\Omega^\vee((v_1, i\alpha_1), \ldots, (v_n, i\alpha_n)) = i^{-n} \int_L (-\iota_{v_1}\omega + \frac{i\alpha_1}{2\pi}) \wedge \cdots \wedge (-\iota_{v_n}\omega + \frac{i\alpha_n}{2\pi}) \tag{3}$$

After normalizing $\int_L \Omega = 1$, we obtained a compatible Kähler form

$$\omega^\vee((v_1, i\alpha_1), (v_2, i\alpha_2)) = \frac{1}{2\pi} \int_L \alpha_2 \wedge (\iota_{v_1} \mathrm{Im}\ \Omega) - \alpha_1 \wedge (\iota_{v_2} \mathrm{Im}\ \Omega) \tag{4}$$

Now, let B be the set of special Lagrangian tori, $\pi^\vee : M \to B, (L, \nabla) \to L$ a special Lagrangian torus fibration (with torus fiber $\{(0, i\alpha)\}$) "dual to $\pi : X \to B$". Note that $\pi^\vee$ has a zero section $\{(L, d)\}$ which is a special Lagrangian, and has complex conjugation $(L, \nabla) \leftrightarrow (L, \nabla^*)$.

Example. As usual, let $T^2 = \mathbb{C}/\mathbb{Z} + i\rho\mathbb{Z}, \Omega = dz, \omega = \frac{\lambda}{\rho} dx \wedge dy, \int_{T^2} \omega = \lambda$. L is special Lagrangian $\Leftrightarrow$ Im $dz_{|}L = 0 \Leftrightarrow L$ is parallel to the real axis. We have a fibration $T^2 \xrightarrow{\pi} S^1 = \mathbb{R}/\rho\mathbb{Z}, (x, y) \mapsto y$, with fibers $L_t = \{y = t\}$, inducing a complex affine structure with affine coordinate y (= Im Ω on the arc from L_0 to L), size$(S^1) = \rho$, and a symplectic affine structure $\frac{\lambda}{\rho} y$ (= the symplectic area swept), size$(S^1) = \lambda$. On the mirror $M = \{(L, \nabla)\} \in \mathbb{R}/\rho\mathbb{Z}$, the holomorphic coordinate for $J^\vee$ is $\exp(-2\pi\frac{\lambda}{\rho} y)e^{i\theta}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}, \nabla = d + i\theta dx$. Or, taking $\frac{1}{2\pi i}$ log), $z^\vee = \frac{\theta}{2\pi} + i\frac{\lambda}{\rho} y \in \mathbb{C}/\mathbb{Z} + i\lambda\mathbb{Z}$. Furthermore $\Omega^\vee = dz^\vee$, $\omega^\vee = \frac{1}{2\pi} d\theta \wedge dy$. Our SYZ transformation exchanges Lagrangian sections of π and flat connections with a connection on a holomorphic line bundle. Explicitly, set $L = \{x = f(y)\}, f :$

$\mathbb{R}/\rho\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, with flat connection $\nabla = d + ih(y)dy, h : \mathbb{R}/\rho\mathbb{Z} \to \mathbb{R}$. We build a Hermitian connection $\check{\nabla} = d+if(y)d\theta+ih(y)dy$ on a localy trivialized Hermitian line bundle $\mathcal{L}$. Note that changing the trivialization by $e^{i\theta}$ changes the connection form by $id\theta$, i.e. $f \leftrightarrow f+1$, glue $y = 0$ to $y = \rho$ by $e^{i\deg\,(f)\theta}$. Furthermore, $\deg\,(\mathcal{L}) = \deg\,(f : S^1 \to S^1)$. We have a holomorhic structure $\overline{\partial}^{\check{\nabla}} = \check{\nabla}^{0,1}$.

In higher-dimensional tori, we have $L = \{x = f(y)\}$ Lagrangian, $f : \mathbb{R}^n/\Lambda \to \mathbb{R}^n/\mathbb{Z}^n$, $\nabla = d + i\sum_j h_j(y)dy_j, h : \mathbb{R}^n/\Lambda \to \mathbb{R}^n$ on the one side, $\check{\nabla} = d + i\sum_j f_j(y)d\theta_j + i\sum_j h_j(y)dy_j$, which is holomorphic $\Leftrightarrow$ the curvature is $(1,1)/J^\vee$. Set

$$F = i\sum_{j,k}\frac{\partial f_j}{\partial y_k}dy_k \wedge d\theta_j + i\sum_{j,k}\frac{\partial h_j}{\partial y_k}dy_k \wedge dy_j \tag{5}$$

Then $J^\vee$ exchanges dy_k and $d\theta_k$ up to canonical scaling, and is holomorphic $\Leftrightarrow$

- $\frac{\partial f_j}{\partial y_k} = \frac{\partial f_k}{\partial y_j}$ for $\sum f_j dy_j$ a closed 1-form on $\mathbb{R}^n/\Lambda$ ($\Leftrightarrow L$ Lagrangian),
- $\frac{\partial h_j}{\partial y_k} = \frac{\partial h_k}{\partial y_j}$ for $\sum h_j dy_j$ a closed 1-form ($\Leftrightarrow \nabla$ is flat).

Example. Let X be a K3 surface, namely a simply connected complex surface with $K_X \cong \mathcal{O}_X$, e.g. a 4-dimensional hypersurface $\{P_4(x_0, \ldots, x_3) = 0\} \subset \mathbb{CP}^3$ for P_4 a homogeneous polynomial in degree 4, or a double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\{z^2 = P_{4,4}((x_0, x_1), (y_0, y_1))\} \subset \mathrm{Tot}(\mathcal{O}(2, 2))$ with Hodge diamond

$$\begin{matrix} 1 & 0 & 1 \\ \\ 0 & 20 & 0 \\ \\ 1 & 0 & 1 \end{matrix} \tag{6}$$

Any K3 surface is *hyperkähler*, i.e. there are three complex structures $I, J, K = IJ = -JI$ inducing three Kähler forms $\omega_I, \omega_J, \omega_K$ for the same Kähler metric g. The idea is the following: given $I, [\omega_I]$, Yau's theorem gives a Ricci-flat Kähler metric g, and we obtain a holomorphic volume form $\Omega_I \in \Omega^{2,0}$ with $|\Omega_I|_g = 1, \Omega_U = \omega_J + i\omega_K$, where ω_I is $(1,1)$ for I, $\omega_J = \mathrm{Re}\ \Omega_I, \omega_K = \mathrm{Im}\ \Omega_I$ $(2,0)+(0,2)$ for I are pointwise orthonormal self-dual 2-forms which are covariantly constant.

Some (not all) K3 surfaces admit fibrations by elliptic curves over spheres, typically with 24 nodal singular fibers. For instance, given a double coordinate of $\mathbb{CP}^1 \times \mathbb{CP}^1$, we project to a $\mathbb{CP}^1$ factor, and observe that the fibers are double covers of $\mathbb{CP}^1$ branched at four points. Now, assume we have one of these with a holomorphic section. The fibers will be I-complex curves, and thus special Lagrangian for $(\omega_J, \Omega_J = w_K + i\omega_I), (\omega_K, \Omega_K = \omega_I + i\omega_J)$ (they are calibrated by ω_I, which is $(1,1)$ for I so ω_J, ω_K vanish). Mirror symmetry corresponds these latter two structures.