

MIRROR SYMMETRY: LECTURE 21

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1. SYZ CONJECTURE (CNTD.)

Question: how does one build a mirror \check{X} of a given Calabi-Yau X ? Recall that homological mirror symmetry says that $D^b\text{Coh}(\check{X}) \cong D^\pi\text{Fuk}(X)$, and points $p \in \check{X}$ correspond to skyscraper sheaves $\mathcal{O}_p \in D^b\text{Coh}(\check{X})$ and $\mathcal{L}_p \in D^\pi\text{Fuk}(X)$. Regarding \check{X} as the moduli space of skyscraper sheaves in $D^b\text{Coh}(\check{X})$ as well as a moduli space of certain objects of $D^\pi\text{Fuk}(X)$, the question reduces to understanding exactly what are these certain objects. Recall that $HF^*(\mathcal{L}_p, \mathcal{L}_p) \cong \text{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) = H^*(T^n, \mathbb{C})$.

Conjecture 1. *Generic points of \check{X} parameterize isomorphism classes of (L, ∇) , $L \subset X$ a Lagrangian torus and ∇ a flat $U(1)$ -connection on $\underline{\mathbb{C}} \rightarrow L$ (corresponding to elements of $\text{Hom}(\pi_1 L, U(1))$).*

Definition 1. A special Lagrangian submanifold is one with $\text{Im}(\Omega|_L) = 0$.

Conjecture 2 (SYZ). X, \check{X} carry dual fibrations by special Lagrangian tori

$$(1) \quad \begin{array}{ccc} T^n & \longrightarrow & X \\ \downarrow \pi & & \downarrow \pi^\vee \\ B & & B \end{array} \quad \check{T}^n \longrightarrow \check{X}, \quad \check{T} = \text{Hom}(\pi_1 T, U(1))$$

i.e. $\check{X} = \{(L, \nabla), L \text{ fiber of } \pi, \nabla \in \text{hom}(\pi_1 T, U(1))\}$.

Remark. Warnings: constructing special Lagrangian fibrations is hard/impossible (... , Joyce,..., Haase-Zharkov, Gross-Siebert,...). They should come from LCSL degenerations, and different LSCL degenerations give different special Lagrangian fibrations and thus different mirrors. Furthermore, special Lagrangian fibrations will typically have singular fibers, so the dual fibration is not well defined. The issue here is “instanton corrections”.

1.1. Special Lagrangian submanifolds. Let (X, ω, J) be a Kähler manifold, $g = \omega(\cdot, J\cdot)$ the Kähler metric, $\Omega \in \Omega^{n,0}$ a holomorphic volume form. It is *strictly Calabi-Yau* if g is Ricci-flat, i.e. $|\Omega|_g$ is constant, or $\Omega \wedge \bar{\Omega} = c(n)\omega^n$, or $\nabla\Omega = 0$, or $\text{hol}_g \subseteq SU(n)$. It is *almost Calabi-Yau* if $|\Omega|_g = \psi \in C^\infty(X, \mathbb{R}_+)$, $\Omega \wedge \bar{\Omega} = c(n)\psi^2\omega^n$.

Proposition 1. *If $L \subset X$ is Lagrangian, $\Omega|_L \in \Omega^n(L, \mathbb{C})$ is $e^{i\phi}\psi\text{vol}_g|_L$ with $e^{i\phi} : L \rightarrow S^1$ is a phase function.*

Proof. Linear algebra! At $p \in L$, $(T_p X, \omega_p, J_p, T_p L) \cong (\mathbb{C}^n, \omega_0, J_0, \mathbb{R}^n)$, and

$$(2) \quad \Omega_p|_{\mathbb{R}^n} = e^{i\phi(p)}\psi(p)dz_1 \wedge \cdots \wedge dz_n|_{\mathbb{R}^n} = e^{i\phi}\psi dx_1 \wedge \cdots \wedge dx_n$$

□

We say that L is *special* if $e^{i\phi} : L \rightarrow S^1$ is constant. Then $\int_L \Omega \in e^{i\phi}\mathbb{R}_+$. Given $[L] \in H_n(X)$, we normalize Ω s.t. $\int_L \Omega = 1$, $\int_L \Omega \in \mathbb{R}_+$. Then this definition of specialness is equivalent to our previous one, i.e. $\text{Im } \Omega|_L = 0$, and $\text{Re } \Omega|_L = \psi\text{vol}_g|_L$.

Remark. In the case of strictly Calabi-Yau manifolds, special Lagrangians are calibrated and hence absolutely volume-minimizing in their homology class. For any n -plane Π , $\text{Re } \Omega|_\Pi \leq \text{vol}_g|_\Pi$, with equality iff Π is special Lagrangian. Thus,

$$(3) \quad [\text{Re } \Omega] \cdot [L] = \int_L \text{Re } \Omega \leq \int_L \text{vol}_g = \text{vol}(L)$$

with equality again if L is special Lagrangian.

Remark. Since $c_1(TX) = 0$, \exists a global \mathbb{Z} -cover of the Lagrangian Grassmannian of X . We can describe a graded Lagrangian plane as a Lagrangian plane $\Pi \subset TX$ equipped with a real lift $\phi \in \mathbb{R}$ of the phase. For an oriented Lagrangian submanifold $L \subset X$, $e^{i\phi} : L \rightarrow S^1$ might not lift to $\phi : L \rightarrow \mathbb{R}$, and the obstruction is a homotopy class in $[L, S^1] = H^1(L, \mathbb{Z})$: up to a factor of 2, this is the Maslov class μ_L . For L a special Lagrangian, $\mu_L = 0$, graded lifts exists, and HF can be \mathbb{Z} -graded.

1.2. Deformations of special Lagrangians. Let $v \in C^\infty(NL)$ be a normal vector field, $\phi_t = \exp(tv)$, $L_t = \phi_t(L)$. One may ask when L_t is special Lagrangian. It is Lagrangian if $\omega|_{L_t} = 0$, i.e. $\phi_t^* \omega = 0$. To first order,

$$(4) \quad \frac{d}{dt}(\phi_t^* \omega)|_{t=0} = (L_v \omega)|_L = (d\iota_v \omega)|_L$$

so $\beta = -\iota_v \omega \in \Omega^1(L, \mathbb{R})$ should be closed. For specialness, need $\text{Im } \Omega|_{L_t} = 0$, i.e. $\phi_t^*(\text{Im } \Omega) = 0$. Again, to first order,

$$(5) \quad \frac{d}{dt}(\phi_t^* \text{Im } \Omega)|_{t=0} = (L_v \text{Im } \Omega)|_L = (d\iota_v \text{Im } \Omega)|_L$$

and $\tilde{\beta} = \iota_v \text{Im } \Omega \in \Omega^{n-1}(L, \mathbb{R})$ should be closed.

Now, in the standard metric on $T_p L \cong \mathbb{R}^n \subset \mathbb{C}^n \cong T_p X$, $\Omega_p = \psi dz_1 \wedge \cdots \wedge dz_n$. Setting $v = \sum a_i \frac{\partial}{\partial y_i}$ gives $\beta = -\iota_v \omega_0 = \sum a_i dx_i$,

$$(6) \quad \tilde{\beta} = \iota_v \text{Im } \Omega|_L = \sum a_i (-1)^{i-1} \psi dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n = \psi \cdot * \beta$$

In the strict Calabi-Yau case, $\tilde{\beta} = *\beta$ so $d\beta = d\tilde{\beta} = 0 \Leftrightarrow \beta$ is harmonic.

Proposition 2. *First order deformations of special Lagrangian L in a strict (resp. almost) Calabi-Yau manifold are given by $\mathcal{H}^1(L, \mathbb{R})$ (resp. $\mathcal{H}_\psi^1(L, \mathbb{R})$), where*

$$(7) \quad H_\psi^1(L, \mathbb{R}) = \{\beta \in \Omega^1(L, \mathbb{R}) \mid d\beta = 0, d^*(\psi\beta) = 0\}$$

It is still true that $\mathcal{H}_\psi^1(L, \mathbb{R}) \cong H^1(L, \mathbb{R})$.

The idea is to redo the Hodge decomposition theorem with

$$(8) \quad \Omega^1 \xrightarrow{(d, \psi^{-1}d^*\psi)} \Omega^0 \oplus \Omega^2$$

Or, if $n \neq 2$, ψ -harmonicity for g is equivalent to harmonicity for $\psi^{\frac{2}{n-2}}g$.

Theorem 1 (McLean, Joyce). *Deformations of special Lagrangians are unobstructed, i.e. the moduli space of special Lagrangians is a smooth manifold B with $T_L B \cong \mathcal{H}_\psi^1(L, \mathbb{R}) \cong H^1(L, \mathbb{R})$.*

Proof. Locally near L , deformations correspond to normal vector fields via the exponential map. Consider the Banach bundle $\mathcal{E} \rightarrow U \subset W^{k,p}(L, NL)$ with fiber at v given by

$$(9) \quad \mathcal{E}_v = W^{k-1,p}(L, \bigwedge^2(T^*L)) \oplus W^{k-1,p}(L, \bigwedge^n T^*L)$$

We have a section of \mathcal{E} given by

$$(10) \quad s = (\exp(v)^*\omega, \exp(v)^*\text{Im } \Omega)$$

which is closed, and even exact. Then $B = s^{-1}(0)$. Let $\mathcal{F}\{\mathcal{E}\}$ be the Banach subbundle of exact forms. Then s is a Fredholm section of \mathcal{F} . Let $\omega^\# : NL \xrightarrow{\sim} T^*L$ be the map $v \mapsto -\iota_v \omega$, we have that

$$(11) \quad ds(0) \circ (\omega^\#)^{-1} : \beta \mapsto (-d\beta, d(\psi \cdot *\beta))$$

is surjective and $s^{-1}(0)$ is smooth. □