MIRROR SYMMETRY: LECTURE 14

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0.1. **Lagrangian Floer Homology (contd).** Let (M, ω) be a symplectic manifold, L_0, L_1 compact Lagrangian submanifolds intersecting transversely. Recall that the complexes $CF(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$ carry a differential m_1, product m_2, and higher operations

$$CF^*(L_0, L_1) \otimes \cdots \otimes CF^*(L_{k-1}, L_k) \stackrel{m_k}{\to} CF^*(L_0, L_k)[2-k] \tag{1}$$

We looked at J-holomorphic maps u from disks D^2 with marked boundary points to disks in the manifold between $L_0, \dots, L_k$ with $u(z_0) = q \in L_0 \cap L_k, u(z_i) = p_i \in L_{i-1} \cap L_i$. We find that the expected dimension of our manifold $\mathcal{M}(p_1, \dots, p_k, q, [u], J)$ is deg $q -$ (deg $p_1 + \cdots$ deg $p_k) + k - 2$. Assuming transversality,

$$m_k(p_k, \dots, p_1) = \sum_{\substack{q \in L_0 \cap L_k \\ \text{ind}([u]) = 0}} (\#\mathcal{M}(p_1, \dots, p_k, q, [u], J)) T^{\omega(u)} q \tag{2}$$

By looking at the ∂ (1-dimensional moduli space), we obtained the A_∞ relations:

Proposition 1. *Assuming no bubbling of disks and spheres,* $\forall\, m \geq 1, (p_1, \dots, p_m)$, $p_i \in L_{i-1} \cap L_i$,

$$\sum_{\substack{k, \ell \geq 1 \\ k + \ell = m+1 \\ 0 \leq j \leq \ell - 1}} (-1)^* m_\ell(p_m, \dots, p_{j+k+1}, m_k(p_{j+k}, \dots, p_{j+1}), p_j, \dots, p_1) = 0 \tag{3}$$

where $* =$ deg $(p_1) + \cdots +$ deg $(p_j) + j$.

This implies that m_1 is a differential, m_2 satisfies the Leibniz rule, and m_2 is associative up to homotopy given by m_3 (i.e. it is associative in HF^*).

Definition 1. *An A_∞ category is a linear "category" where morphism spaces are equipped with algebraic operations $(m_k)_{k \geq 1}$ satisfying the A_∞-relations (those defined above).*

In our case, we have the following categories:

- A Fukaya category is any of a number of A_∞ categories whose objects are Lagrangian submanifolds (with extra data), the morphisms are Floer complexes, and the algebraic operations are as above.
- So far we only have an 'A_∞-precategory" because the homomorphisms have only been defined for transverse pairs of objects.
- At the homology level, we can also define the *Donaldson-(Fukaya category)* whose homomorphisms are the cohomologies HF, so that composition is automatically associative. This is technically easier, but we lose some information that we need for mirror symmetry.
- We eventually want to define our Fukaya category to be over $\mathbb{C}$, rather than over the Novikov ring. So far, we have counted disks with weights $T^{\omega(u)} \in \Lambda$, and Gromov compactness tells us that there are only finitely many contributions below a certain area. That is, the sums may be infinite, but they converge in the Novikov ring. Physicists usually write the terms as $e^{-2\pi\omega(u)} \in \mathbb{R}$ instead of $T^{\omega(u)}$, and hope for convergence. Changing the value of T is equivalent to rescaling the symplectic form, i.e. working over Λ is equivalent to working with a family $M, (\omega_t = t\omega)$, with $T = e^{-2\pi t}$. We thus work near the large volume limit $t \to \infty$ and compute Floer homologies for all t simultaneously. We call this the "convergent power series" Floer homology: even when defined, this is often not Hamiltonian isotopy invariant.
- For Lagrangians L_i equipped with $(E_i, \nabla_i) \to L_i$ complex vector bundles with flat (unitary) connections. We think of these as local systems of coefficients on our Lagrangians. We define an associated complex with twisted coefficients:

$$CF((L_0, E_0, \nabla_0), (L_1, E_1, \nabla_1)) = \oplus_{p \in L_0 \cap L_1} \mathrm{Hom}((E_0)_p, (E_1)_p) \otimes \Lambda \tag{4}$$

for L_0, L_1 transverse. Then given $p_1, \ldots, p_k, p_i \in L_{i-1} \cap L_i, w_1, \ldots, w_k, w_i \in \mathrm{Hom}((E_{i-1})_{p_i}, (E_i)_{p_i})$, we let

$$m_k(w_k, \ldots, w_1) = \sum_{\substack{q \in L_0 \cap L_k \\ \mathrm{ind}([u]) = 0}} (\#\mathcal{M}(p_1, \ldots, p_k, q, [u], J)) T^{\omega(u)} \mathcal{P}_{[\partial u]}(w_k, \ldots, w_1) \tag{5}$$

where $\mathcal{P}_{[\partial u]}(w_k, \ldots, w_1) \in \mathrm{Hom}((E_0)_q, (E_k)_q)$ is defined by

$$\mathcal{P}_{[\partial u]}(w_k, \ldots, w_1) = \gamma_k \circ w_k \circ \gamma_{k-1} \circ \cdots \circ \gamma_1 \circ w_1 \circ \gamma_0 \tag{6}$$

where parallel transport along ∂u from $q \to p_1$ gives $\gamma_0 \in \mathrm{Hom}((E_0)_q, (E_0)_{p_1})$, and similarly parallel transport from $p_i \to p_{i+1}$ using ∇_i gives $\gamma_i \in \mathrm{Hom}((E_i)_{p_i}, (E_i)_{p_{i+1}})$. For ∇_i flat, this only depends on $[\partial u]$. In particular, if E_i is the topologically trivial line bundle $\mathbb{C} \times L_i$ and ∇_i is a flat $U(1)$

connection (up to gauge equivalence), $\nabla_i = d + iA_i$ for A_i a closed 1-form, this encodes the data of holonomies $\pi_1(L_i) \to U(1)$. Then, using trivializations, we get $CF = \Lambda_{\mathbb{C}}^{|L_0 \cap L_1|}$ with generators $p, w = \mathrm{id} \ : E_{0_p} \to E_{1_p}$ and m_k counts disks with weight $T^{\omega(u)} \cdot \mathrm{hol}(\partial u)$, where

$$\mathrm{hol}(\partial u) = \exp\left(i \sum_{j=0}^{k} \int_{\partial u_j} A_j \right) \tag{7}$$

is the holonomy of parallel transport.

We can now construct our first iteration of the Fukaya category:

- The objects are $\mathcal{L} = (L, E, \nabla)$, where L is a compact spin Lagrangian ($\mathbb{Z}$-graded: $\mu_L = 0$ with grading data) and (E, ∇) a flat hermitian vector bundle.
- The morphisms for transverse $\mathcal{L}_0, \mathcal{L}_1$ is given by $\hom(\mathcal{L}_0, \mathcal{L}_1) = CF^*$.

Issues:

(1) What if L_0 is not transverse to L_1 (in particular, if $L_0 = L_1$)?
(2) What if L bounds disks?

For the first problem, see Seidel's book: the idea is to use a Hamiltonian perturbation ϕ_H to get L_1 to be transverse to L_0, and define $CF^*(L_0, L_1)$ to be generated by $L_0 \cap \phi_H(L_1)$ (the vector bundles carry without change). We perturb all the $\overline{\partial}$-equations by suitable terms: in the strip-like ends, we have $\frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} + X_H(u)) = 0$ for $H = H(L_{i-1}, L_i)$. We need a procedure to associate to (L, L') a Hamltonian $H(L, L')$, and to a sequence $L_0, \ldots, L_k$ some compatible perturbation data, and further to show that different choices give equivalent A_∞-categories. Note that this will not be strictly unital, and will only get a homology unit.

Alternatively, one can use "Morse-Bott" Floer theory (e.g. FOOO). We define $CF^*(L, L) = C_*(L; \Lambda)$ to be the space of singular chains on L: when defining the operations m_k, instead of strip-like ends, we have a marked point z on the boundary such that when evaluating at z, and require $u(z)$ to be in the chain. For instance, in the product m_2, one considers disks with boundary points z_0, z_1, z_2 with three evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,3}(M, L; J, \beta) \to L$,

$$m_2(C_2, C_1) = \sum_{\beta \in \pi_2(X, L)} T^{\omega(\beta)} (ev_0)_*([\overline{\mathcal{M}}_{0,3}(M, L; J, \beta)] \cap ev_1^* C_1 \cap ev_2^* C_2) \tag{8}$$

For the class $\beta = 0$, we find that the contribution of constant disks gives the intersection product on $C_*(L)$. The higher m_k follow similarly, though m_1 does not allow $\beta = 0$ and adds the classical ∂C instead. More generally, if $L_0 \cap L_1$ have a "clean intersection" (i.e. $L_0 \cap L_1$ is smooth), then we set $CF^*(L_0, L_1) = C_*(L_0 \cap L_1; \Lambda)$.