# MIRROR SYMMETRY: LECTURE 11

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0.1. **Lagrangian Floer Homology (contd).** Let $(M, \omega)$ be a symplectic manifold, $L_0, L_1$ compact Lagrangian submanifolds. Formally, Floer homology is Morse theory for the action functional on the path space $\mathcal{P}(L_0, L_1)$, which has as critical points the constant paths. More precisely, the actual functional is a map $\tilde{A} : \tilde{\mathcal{P}}(L_0, L_1) \to \mathbb{R}$, where $\tilde{\mathcal{P}}(L_0, L_1)$ is the universal cover of the path space, i.e. pairs $(\gamma, [u])$ where $\gamma$ is a path between $L_0$ and $L_1$ and $[u]$ is a homotopy between $\gamma$ and some fixed base path $*$. Then $\mathcal{A}(\gamma, [u]) = \int u^*\omega$, and for $v$ a vector field along $\gamma$,

$$d\mathcal{A}(\gamma) \cdot v = \int_{[0,1]} \omega(\dot{\gamma}, v) dt = \int_{[0,1]} g(J\dot{\gamma}, v) dt = \langle J\dot{\gamma}, v \rangle_{L^2} \tag{1}$$

The critical points are contant paths $\dot{\gamma} = 0$, and the gradient flow lines are $J$-holomorphic curves $\frac{\partial \gamma}{\partial s} = -J\dot{\gamma}$.

However, no one has managed to run this Morse theory rigorously. The actual setup assumes $L_0, L_1$ are transverse, and as before, define the Novikov ring as $\Lambda = \{\sum a_i T^{\lambda_i} \mid \lambda_i \to \infty\}$ and the *Floer complex* $CF(L_0, L_1)$ as the free $\Lambda$-module $\Lambda^{|L_0 \cap L_1|}$ generated by $L_0 \cap L_1$. We look at $u : \mathbb{R} \times [0,1] \to M$ equipped with a compatible almost-complex structure $J$ s.t.

- $\overline{\partial}_J u = 0$, or $\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$.
- $u(s, 0) \in L_0, u(s, 1) \in L_1$
- $\lim_{s \to +\infty} u(s,t) = p, \lim_{s \to -\infty} u(s,t) = q$ for $\{p, q\} \subset L_0 \cap L_1$
- $E(u) = \int u^*\omega = \int \int_{\mathbb{R} \times [0,1]} \left| \frac{\partial u}{\partial s} \right|^2 ds dt < \infty$.

We consider the space of solutions $\mathcal{M}(p, q, [u], J)$ for fixed $p, q \in L_0 \cap L_1$, $[u]$ a homotopy class as above, and $J$ a given almost-complex structure. The above problem is a Fredholm problem, and the expected dimension of $\mathcal{M} = \mathrm{ind}(\overline{\partial}_J)$ is called the *Maslov index*. The Maslov index comes from $\pi_1(\bigwedge \mathrm{Gr}) = \mathbb{Z}$. Explicitly, let $L_0, L_1(t)_{t \in [0,1]}$ be Lagrangian subspaces of $\mathbb{R}^{2n}$ s.t. $L_1(0), L_1(1)$ intersect $L_0$ transversely. The Maslov index of $(L_1(t); L_0)$ is the number of times that $L_1(t)$ is non-transverse to $L_0$ with mutlipliticities and signs. For instance, for $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$, $L_1(t) = (e^{i\theta_1(t)}\mathbb{R}) \times \cdots \times (e^{i\theta_n(t)}\mathbb{R})$ with all $\theta_i$'s increasing past 0, the Maslov index is $n$. In general, given a homotopy $u$, we can trivialize $u^*TM$, and $u^*|_{\mathbb{R} \times 0}(TL_0), u^*|_{\mathbb{R} \times 1}(TL_1)$ are 2 paths of Lagrangian subspaces. We can trivialize so that $TL_0$ remains constant, and $\mathrm{ind}(u)$ is the Maslov index of the path $TL_1$ relative to $TL_0$ as one goes from $p$ to $q$.

Now, we want to define

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ \phi \in \pi_2(M, L_0 \cup L_1) \\ \mathrm{ind}(\phi) = 1}} \#(\mathcal{M}(p,q,\phi,J)/\mathbb{R}) T^{\omega(\phi)} \cdot q \tag{2}$$

The issues that arise are: transversality, compactness and bubbling, the orientation of $\mathcal{M}$, and whether $\partial^2 = 0$.

**Theorem 1.** *If $[\omega] \cdot \pi_2(M) = 0$ and $[\omega] \cdot \pi_2(M, L_i) = 0$, then $\partial$ is well-defined, $\partial^2 = 0$, and $HF(L_0, L_1) = H^*(CF, \partial)$ is independent of the chosen $J$ and invariant under Hamiltonian isotopies of $L_0$ and/or $L_1$.*

**Corollary 1.** *If $[\omega] \cdot \pi_2(M, L) = 0$ and $\psi$ is a Hamiltonian diffeomorphism s.t. $\psi(L), L$ are transverse, $\#(\psi(L) \cap L) \geq \sum b_i(L)$.*

This is a special case of Arnold's conjecture: the rough idea is that $H^*(L) \cong HF(L, \psi(L))$ and $\mathrm{rk}\, CF \geq \mathrm{rk}\, HF$.

*Example.* Consider $T^*S_1 \cong \mathbb{R} \times S^1$, with $L_0 = \{(0,\theta) \,|\, \theta \in S^1 = [0, 2\pi)\}, L_1 = \{(a \sin\theta + b, \theta)\}$. Then $L_0 \cap L_1 = \{p, q\}$, and the region between them decomposes into disks $u, v$. Then $CF(L_0, L_1) = \bigwedge p \oplus \bigwedge q, \partial(p) = (T^{area(u)} - T^{area(v)})q, \partial(q) = 0$. In this case ($c_1(TM) = 0$, as is the Maslov class of $L_i$), $\exists$ a $\mathbb{Z}$ grading on $CF$ (because the index is independent of $[u]$), e.g. $\deg p = 0, \deg q = 1$. We have two cases:

- if $area(u) = area(v)$, then $\partial = 0, HF(L_0, L_1) \cong H^*(S^1, \Lambda)$.
- if $area(u) \neq area(v)$, then $HF(L_0, L_1) = 0$.

Return to our issues, one can achieve transversality for simple maps by picking $J$ generic, but for multiply covered maps, we need sophisticated techniques such as domain-dependent $J$, multivalued perturbations, virtual cycles, or Kuranishi structures. To obtain an orientation of the moduli space, we need auxiliary data, e.g. a spin structure on $L_0, L_1$. For compactness, the Gromov compactness theorem implies that, given an energy bound, compactness holds after adding limiting configurations. There are three types of phenomena:

- Bubbling of spheres: if $|du_n| \to \infty$ at an interior point, the resulting limit is a spherical bubble. The treatment is the same as in Gromov-Witten invariants, and in good cases (if transversality is achieved), the congurations with sphere bubbles have real codimension $\geq 2$ in $\overline{\mathcal{M}}$.
- Bubbling of disks: if $|du_n| \to \infty$ at a boundary point, the resulting limit is a disk bubble at the boundary. Even assuming transversality, the space of these will have real codimension 1 in $\overline{\mathcal{M}}$.

- Breaking of strips: if energy escapes towards $s \to \pm\infty$, i.e. reparameterizing $u_n(\cdot - \delta_n, \cdot)$ for $|\delta_n| \to \infty$ gives different limits, the resulting limit is a sequence of holomorphic strips (that is, what was a single holomorphic strip with progressively thinning "necks" becomes several separate strips).

Finally, we want to have $\partial^2 = 0$. Asuming no bubbling, we consider $\mathcal{M}(p, q, \phi, J)/\mathbb{R}$ for $J$ generic, $\phi \in \pi_2, \mathrm{ind}(\phi) = 2$. We expect a one-dimensional manifold, which is compactified by adding broken trajectories, i.e.

$$\sqcup_{\substack{r \in L_0 \cap L_1 \\ \phi_1 \# \phi_2 = \phi}} (\mathcal{M}(p, r, \phi_1, J)/\mathbb{R}) \times (\mathcal{M}(p, r, \phi_2, J)/\mathbb{R}) \tag{3}$$

The gluing theorem states that the resulting $\overline{\mathcal{M}(p, q, \phi, J)/\mathbb{R}}$ is a manifold with boundary. Now, the number of ends of a compact oriented 1-manifold is 0, and thus so are the contributions to the coefficients of $T^{\omega(\phi)} q$ in $\partial^2(p)$.