3 Integration on manifolds, Lecture 3

In this section we will show how to integrate densities over manifolds. First, however, we will have to explain how to integrate densities over open subsets, U, of $\mathbb{R}^n$. Recall that if φ is in $\mathcal{D}^\infty(U)$ it can be written as a product, $\sigma = \psi\sigma_{\mathrm{Leb}}$, where ψ is in $\mathcal{C}^\infty(U)$. We will say that σ is integrable over U if ψ is integrable over U, and will define the integral of σ over U to be the usual Riemann integral

$$\int_U \sigma = \int_U \psi\, dx\,. \tag{3.1}$$

The advantage of using "density" notation for this integral is that it makes the change of variables formula more transparent. Namely if U_1 is an open subset of $\mathbb{R}^n$ and $f : U_1 \to U$ a diffeomorphism, then by (2.4) $f^*\sigma = \psi_1\sigma_{\mathrm{Leb}}$ where

$$\psi_1 = \psi \circ f \left|\det \left[\frac{\partial f_i}{\partial x_j}\right]\right| \tag{3.2}$$

and hence by the change of variables formula[1] ψ_1 is integrable over U_1 and

$$\int_{U_1} \psi_1\, dx = \int_U \psi\, dx\,.$$

Thus using density notation the change of variables formula takes the much simpler form

$$\int_{U_1} f^*\sigma = \int_U \sigma\,. \tag{3.3}$$

Now let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold. Our goal below will be to define the integral

$$\int_W \sigma \tag{3.4}$$

where W is an open subset of X and σ is a compactly supported $\mathcal{C}^\infty$ density. We'll first show how to define this integral when the support of σ is contained in a "parametrizable" open subset of X and then, using partition of unity argument, define it in general.

Definition 1. *An open subset, U, of X is* parametrizable *if there exists an open set, U_0, in $\mathbb{R}^n$ and a diffeomorphism, $\varphi_0 : U_0 \to U$.*

In other words "U is parametrizable" means that there exists a parameterization, (U_0, φ_0), of U. It's clear that if U is parametrizable every open subset of U is parametrizable, and, in particular, if U_1 and U_2 are parametrizable, so is $U_1 \cap U_2$. Moreover the definition of manifold says that every point, $p \in X$, is contained in a parametrizable open set.

[1] See Theorem 17.2 in Munkres.

Let σ be an element of $\mathcal{D}_0^\infty(X)$ whose support is contained in a parametrizable open set U. Picking a parameterization, $\varphi_0 : U_0 \to U$ we will define the integral of σ over W by defining it to be

$$\int_W \sigma = \int_{W_0} \varphi_0^* \sigma \tag{3.5}$$

where $W_0 = \varphi_0^{-1}(W)$. Note that since σ is compactly supported on U, $\varphi_0^*\sigma$ is a product, $\varphi_0^*\sigma = \psi\sigma_{\text{Leb}}$, with ψ in $\mathcal{C}_0^\infty(U_0)$. Hence by Munkres, Theorem 15.2, ψ is integrable over W_0 and hence so is $\varphi^*\sigma$. We will prove

Lemma 1. *The definition (3.5) doesn't depend on the choice of the parameterization,* (U_0, φ_0).

Proof. Let (U_1, φ_1) be another parameterization of U and let $f = \varphi_1^{-1} \circ \varphi_0$. Since φ_0 and φ_1 are diffeomorphisms of U_0 and U_1 onto U f is a diffeomorphism of U_0 onto U_1 with the property

$$\varphi_1 \circ f = \varphi_0 \, . \tag{3.6}$$

In particular if $W_i = \varphi_i^{-1}(W)$, $i = 1, 2$ it follows from (3.6) that f maps W_0 diffeomorphically onto W_1 and from the chain rule it follows that $f^*\varphi_1^*\sigma = \varphi_0^*\sigma$. Hence by (3.3)

$$\int_{W_0} \varphi_0^*\sigma = \int_{W_1} \varphi_1^*\sigma \, . \tag{3.7}$$

In other words (3.5) is unchanged if we substitute (U_1, φ_1) for (U_0, φ_0).

□

From the additivity of the Riemann integral for integrable functions on open subsets of $\mathbb{R}^n$ we also conclude

Lemma 2. *If* $\sigma_i \in \mathcal{D}_0^\infty(X)$, $i = 1, 2$, *is supported on* U

$$\int_W \sigma_1 + \sigma_2 = \int_W \sigma_1 + \int_W \sigma_2$$

and if $\sigma \in \mathcal{D}_0^\infty(X)$ *is supported on* U *and* $c \in \mathbb{R}$

$$\int_W c\sigma = c \int_W \sigma \, .$$

To define the integral (3.4) for arbitrary elements of $\mathcal{D}^\infty(X)$ we will resort to the same partition of unity arguments that we used earlier in the course to define improper integrals of functions over open subsets of $\mathbb{R}^n$. To do so we'll need the following manifold version of Munkres' Theorem 16.3.

Theorem 3. *Let*

$$\mathbb{U} = \{U_\alpha \, , \, \alpha \in \mathcal{I}\} \tag{3.8}$$

be a covering of X *be open subsets. Then there exists a family of functions,* $\rho_i \in \mathcal{C}_0^\infty(X)$, $i = 1, 2, 3, \ldots$, *with the properties* *(a)* $\rho_i \geq 0$.

(b) For every compact set, $C \subseteq X$ *there exists a positive integer* N *such that if* $i > N$, $\operatorname{supp}\, \rho_i \cap C = \emptyset$.

(c) $\sum \rho_i = 1$.

(d) For every i *there exists an* $\alpha \in \mathcal{I}$ *such that* $\operatorname{supp}\, \rho_i \subseteq U_\alpha$.

Remark. *Conditions (a)–(c) say that the* ρ_i*'s are a partition of unity and (d) says that this partition of unity is subordinate to the covering (3.8).*

Proof. To simplify the proof a bit we'll assume that X is a closed subset of $\mathbb{R}^N$. For each U_α choose an open subset, $\mathcal{O}_\alpha$ in $\mathbb{R}^N$ with

$$U_\alpha = \mathcal{O}_\alpha \cap X \tag{3.9}$$

and let $\mathcal{O}$ be the union of the $\mathcal{O}_\alpha$'s. By the theorem in Munkres that we cited above there exists a partition of unity, $\tilde{\rho}_i \in \mathcal{C}_0^\infty(\mathcal{O})$, $i = 1, 2, \ldots$, subordinate to the covering of X by the $\mathcal{O}_\alpha$'s. Let ρ_i be the restriction of $\tilde{\rho}_i$ to X. Since the support of $\tilde{\rho}_i$ is compact and X is closed, the support of ρ_i is compact, so $\rho_i \in \mathcal{C}_0^\infty(X)$ and it's clear that the ρ_i's inherit from the $\tilde{\rho}_i$'s the properties (a)–(d).

□

Now let the covering (3.8) be any covering of X by parametrizable open sets and let $\rho_i \in \mathcal{C}_0^\infty(X)$, $i = 1, 2, \ldots$, be a partition of unity subordinate to this covering. Given $\sigma \in \mathcal{D}_0^\infty(X)$ we will define the integral of σ over W by the sum

$$\sum_{i=1}^{\infty} \int_W \rho_i \sigma \,. \tag{3.10}$$

Note that since each ρ_i is supported in some U_α the individual summands in this sum are well-defined and since the support of σ is compact all but finitely many of these summands are zero by part (b) of Theorem 3. Hence the sum itself is well-defined. Let's show that this sum doesn't depend on the choice of $\mathbb{U}$ and the ρ_i's. Let $\mathbb{U}'$ be another covering of X by parametrizable open sets and ρ'_j, $j = 1, 2, \ldots$, a partition of unity subordinate to $\mathbb{U}'$. Then

$$\begin{aligned} \sum_j \int_W \rho'_j \sigma &= \sum_j \int_W \sum_i \rho'_j \rho_i \sigma \\ &= \sum_j \left(\sum_i \int_W \rho'_j \rho_i \sigma \right) \end{aligned} \tag{3.11}$$

by Lemma 2. Interchanging the orders of summation and resumming with respect to the j's this sum becomes

$$\sum_i \int_W \sum_j \rho'_j \rho_i \sigma$$

or

$$\sum_i \int_W \rho_i \sigma \,.$$

Hence

$$\sum_i \int_W \rho'_j \sigma = \sum_i \int_W \rho_i \sigma \,,$$

so the two sums are the same. Q.E.D.

From (3.10) and Lemma 2 one easily deduces

Proposition 4. *For* $\sigma_i \in \mathcal{D}_0^\infty(X)$, $i = 1, 2$

$$\int_W \sigma_1 + \sigma_2 = \int_W \sigma_1 + \int_W \sigma_2 \tag{3.12}$$

and for $\sigma \in \mathcal{D}_0^\infty(X)$ *and* $c \in \mathbb{R}$

$$\int_W c\sigma = c \int_W \sigma \,. \tag{3.13}$$

In the definition of the integral (3.4) we've allowed W to be an arbitrary open subset of X but required $\sigma \in \mathcal{D}^\infty(X)$ to be compactly supported. This integral is also well-defined if we allow σ to be an arbitrary element of $\mathcal{D}^\infty(X)$ but require the closure of W in X to be compact. To see this, note that under this assumption the sum (3.10) is still a finite sum, so the definition of the integral still makes sense, and the double sum on the right side of (3.11) is still a finite sum so it's still true that the definition of the integral doesn't depend on the choice of partitions of unity. In particular if the closure of W in X is compact we will define the volume of W to be the integral,

$$\mathrm{vol}(W) = \int_W \sigma_{\mathrm{vol}} \,, \tag{3.14}$$

and if X itself is compact we'll define its volume to be the integral

$$\mathrm{vol}(X) = \int_X \sigma_{\mathrm{vol}} \,. \tag{3.15}$$

(For an alternative way of defining the volume of a manifold see Munkres, §22.)

We'll conclude this discussion of integration by proving a manifold version of the change of variables formula (3.3).

Theorem 5. *Let X' and X be n-dimensional manifolds and $f : X' \to X$ a diffeomorphism. If W is an open subset of X and $W' = f^{-1}(W)$*

$$\int_{W'} f^*\sigma = \int_W \sigma \tag{3.16}$$

for all $\sigma \in \mathcal{D}_0^\infty(X)$.

Proof. By (3.11) the integrand of the integral above is a finite sum of $\mathcal{C}^\infty$ densities, each of which is supported on a parametrizable open subset, so we can assume that σ itself as this property. Let V be a parametrizable open set containing the support of σ and let $\varphi_0 : U \to V$ be a parameterization of V. Since f is a diffeomorphism its inverse exists and is a diffeomorphism of X onto X_1. Let $V' = f^{-1}(V)$ and $\varphi_0' = f^{-1} \circ \varphi_0$. Then $\varphi_0' : U \to V'$ is a parameterization of V'. Moreover, $f \circ \varphi_0' = \varphi$ so if $W_0 = \varphi_0^{-1}(W)$ we have

$$W_0 = (\varphi_0)_0^{-1}(f^{-1}(W)) = (\varphi_0')^{-1}(W')$$

and by the chain rule we have

$$\varphi_0^*\sigma \quad = \quad (f \circ \varphi')^*\sigma = (\varphi_0')^* f^*\sigma$$

hence

$$\int_W \sigma \quad = \quad \int_{W_0} \varphi_0^*\sigma = \int_{W_0} (\varphi_0')^*(f^*\sigma) = \int_{W'} f^*\sigma\,.$$

□