Gromov-Witten Invariants and Schubert Polynomials

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based on a joint paper with

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"Quantum Schubert polynomials"

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1. Flag Manifold

$Fl_n = Fl(\mathbb{C}^n)$ flag manifold. Points are

$$U_1 \subset U_2 \subset \cdots \subset U_n = \mathbb{C}^n, \qquad \dim U_i = i.$$

Homomorphism $\alpha : \mathbb{Z}[x_1, \ldots, x_n] \to \mathrm{H}^*(Fl_n, \mathbb{Z})$

$$\alpha : x_i \longmapsto -c_1(\mathcal{E}_i/\mathcal{E}_{i-1}) \in \mathrm{H}^2(Fl_n),$$

where $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathbb{C}^n$ are tautological vector bundles on Fl_n and c_1 is the first Chern class.

Theorem [A. Borel, 1953] The map α induces the isomorphism

$$\boxed{\mathbb{Z}[x_1, \ldots, x_n]/I_n \ \xrightarrow{\sim}\ \mathrm{H}^*(Fl_n, \mathbb{Z})}$$

where $I_n = (e_1, \ldots, e_n)$ is the ideal generated by elementary symmetric polynomials in $x_1, \ldots, x_n$.

Schubert Classes

Another description of $\mathsf{H}^*(Fl_n)$ is based on a decomposition of Fl_n into Schubert cells, labelled by permutations $w \in S_n$

Fix a flag $V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$.
The Schubert variety Ω_w is the set of all flags $U_\bullet \in Fl_n$ such that for all $p, q \in \{1, \ldots, n\}$

$$\dim(U_p \cap V_q) \geq \#\{1 \leq i \leq p, w(i) \geq n+1-q\}$$

Then $\mathrm{codim}_{\mathbb{R}} \Omega_w = 2l(w)$, where $l(w)$ is the length of w.

Schubert class:

$$\sigma_w = [\Omega_w] \in \mathsf{H}_{n(n-1)-2l}(Fl_n) \simeq \mathsf{H}^{2l}(Fl_n)$$

Theorem [Ehresmann, 1934] The classes σ_w, $w \in S_n$, form an additive basis in $\mathsf{H}^*(Fl_n, \mathbb{Z})$.
In particular, $\dim \mathsf{H}^*(Fl_n) = n!$.

Q: How to multiply Schubert classes?

Q': How to express a Schubert class in terms of generators x_i.

Answer (due to Bernstein, Gelfand,Gelfand) can be given in terms of divided differences.

Divided differences

S_n acts on $f \in \mathbb{Z}[x_1, \dots, x_n]$ by

$$w\, f(x_1, \dots, x_n) = f(x_{w^{-1}(1)}, \dots, x_{w^{-1}(n)}).$$

Let $s_i = (i, i+1) \in S_n$ (adjacent transposition).

The divided difference operators are given by

$$\boxed{\partial_i f = \frac{1}{x_i - x_{i+1}}(1 - s_i)\, f}$$

Schubert polynomials

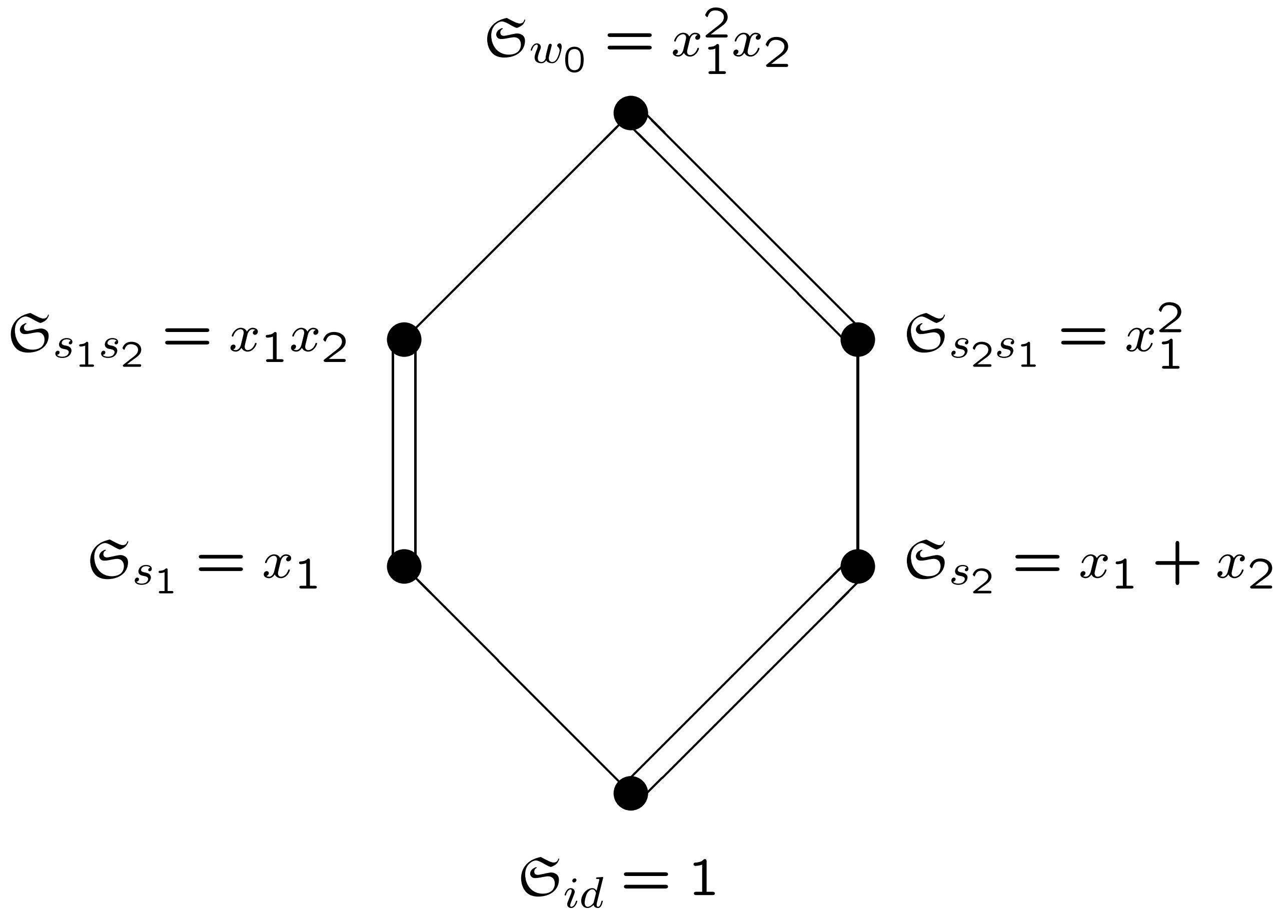
Define the Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$ recursively by

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \dots x_{n-1},$$

(the choice of Lascoux–Schützenberger [1982]) where $w_0 = (n-1, n-2, \dots, 1)$ is the longest permutation in S_n, and

$$\mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w \quad \text{whenever} \quad l(ws_i) = l(w) - 1.$$

Theorem [BGG, 1973] $\mathfrak{S}_w$ represents Schubert class σ_w.



Schubert polynomials for S_3

2. Gromov-Witten Invariants and Quantum Cohomology

(see [Ruan-Tian, Kontsevich-Manin])

Structure constants of the quantum cohomology ring $\mathrm{QH}^*(X)$ are the *Gromov-Witten invariants* (for genus 0).

An algebraic map $f : \mathbb{P}^1 \to Fl_n$ has <u>multidegree</u> $d = (d_1, \ldots, d_{n-1}) \in \mathbb{Z}_+^{n-1}$,
$d_i =$ degree of $f_i : \mathbb{P}^1 \to Fl_n \to Gr(n, i)$.

$\mathcal{M}_d(\mathbb{P}^1, Fl_n) =$ moduli space of such maps.
For $Y \subset Fl_n$, $t \in \mathbb{P}^1$, denote

$$Y(t) = \{f \in \mathcal{M}_d(\mathbb{P}^1, Fl_n) \mid f(t) \in Y\}.$$

<u>Gromov-Witten invariants:</u> Fix $t_1, t_2, t_3 \in \mathbb{P}^1$.

$$\boxed{\langle \sigma_u, \sigma_v, \sigma_w \rangle_d = \# \quad \widetilde{\Omega}_u(t_1) \cap \widetilde{\Omega}_v(t_2) \cap \widetilde{\Omega}_w(t_3)}$$

provided $l(u) + l(v) + l(w) = \dim \mathcal{M}_d(\mathbb{P}^1, Fl_n)$
$\widetilde{\Omega}_u, \widetilde{\Omega}_v, \widetilde{\Omega}_w$ are generic translates of $\Omega_u, \Omega_v, \Omega_w$.

Quantum multiplication

As an abelian group

$\mathsf{QH}^* = \mathsf{QH}^*(Fl_n) = \mathsf{H}^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}]$

Let $*: \mathsf{QH}^* \otimes \mathsf{QH}^* \to \mathsf{QH}^*$ be the $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-linear operation defined by

$$\sigma_u * \sigma_v = \sum_{w \in S_n} \sum_d q^d \langle \sigma_u, \sigma_v, \sigma_{w\,w_0} \rangle_d \, \sigma_w.$$

Then $(\mathsf{QH}^*, *)$ is a commutative and associative algebra called the quantum cohomology ring of Fl_n.

Remark: $\langle \sigma_u, \sigma_v, \sigma_w \rangle_{(0,\ldots,0)}$ is the ordinary intersection number. If we specialize $q_1 = \cdots = q_{n-1} = 0$, the operation $*$ becomes the standard multiplication in $\mathsf{H}^*(Fl_n)$ (the "classical limit").

Quantum analogue of Borel's theorem

Let $E_1, \dots, E_n$ be the quantum elementary polynomials defined to be the coefficients of the characteristic polynomial of the matrix

$$\begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 & 0 \\ -1 & x_2 & q_2 & \cdots & 0 & 0 \\ 0 & -1 & x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n-1} & q_{n-1} \\ 0 & 0 & 0 & \cdots & -1 & x_n \end{pmatrix}$$

Example: $n = 3$

$$\begin{aligned} E_1 &= x_1 + x_2 + x_3, \\ E_2 &= x_1x_2 + x_1x_3 + x_2x_3 + q_1 + q_2, \\ E_3 &= x_1x_2x_3 + q_1x_3 + q_2x_1. \end{aligned}$$

Theorem [Givental, Kim, Ciocan-Fontanine, 1993–1996] There is a canonical isomorphism

$$\boxed{\mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}]/I_n^q \xrightarrow{\sim} \mathrm{QH}^*(Fl_n, \mathbb{Z})}$$

where I_n^q be the ideal generated by $E_1, \dots, E_n$.

3. Main Results

Q: How to multiply Schubert classes in QH*?

Q': How to calculate the Gromov-Witten invariants $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d$?

Q": How to express σ_w in terms of x_i in the ring QH*.

$$\begin{array}{ccc} \mathrm{H}^*(Fl_n) \otimes \mathbb{Z}[q_j] & \cong & \mathbb{Z}[x_i, q_j]/I_n \\ \| & & ? \\ \mathrm{QH}^*(Fl_n) & \cong & \mathbb{Z}[x_i, q_j]/I_n^q \end{array}$$

We will construct the isomorphism

$$\mathbb{Z}[x_i, q_j]/I_n \longrightarrow \mathbb{Z}[x_i, q_j]/I_n^q$$

("quantization map")

Standard elementary monomials

e_i^k = the i^{th} elementary symmetric polynomial in $x_1, \ldots, x_k$ and

E_i^k = the i^{th} quantum elementary polynomial in $x_1, \ldots, x_k$.

For $I = (i_1, i_2, \ldots, i_{n-1})$, $\quad 0 \le i_p \le p$

$$e_I = e_{i_1}^1 e_{i_2}^2 \ldots e_{i_{n-1}}^{n-1},$$

$$E_I = E_{i_1}^1 E_{i_2}^2 \ldots E_{i_{n-1}}^{n-1}$$

Lemma Both $\{e_I\}$ and $\{E_I\}$ are K-liner bases in $K[x_1, x_2, \ldots]$, where $K = \mathbb{Z}[q_1, q_2, \ldots]$.

Quantization map

Define the K-liner map $\psi : K[x_1, x_2, \ldots] \to K[x_1, x_2, \ldots]$ by

$$\boxed{\psi : e_I \longmapsto E_I \text{ for all } I}$$

Remark. ψ induces a map

$$K^n[x_1, \ldots, x_n]/I_n \longrightarrow K^n[x_1, \ldots, x_n]/I_n^q$$

where $K^n = \mathbb{Z}[q_1, \ldots, q_{n-1}]$.

Quantum Schubert polynomials:

Define $$\boxed{\mathfrak{S}_w^q := \psi(\mathfrak{S}_w)}$$

Theorem [FGP] The quantum Schubert polynomial $\mathfrak{S}_w^q$ represents the Schubert class σ_w in

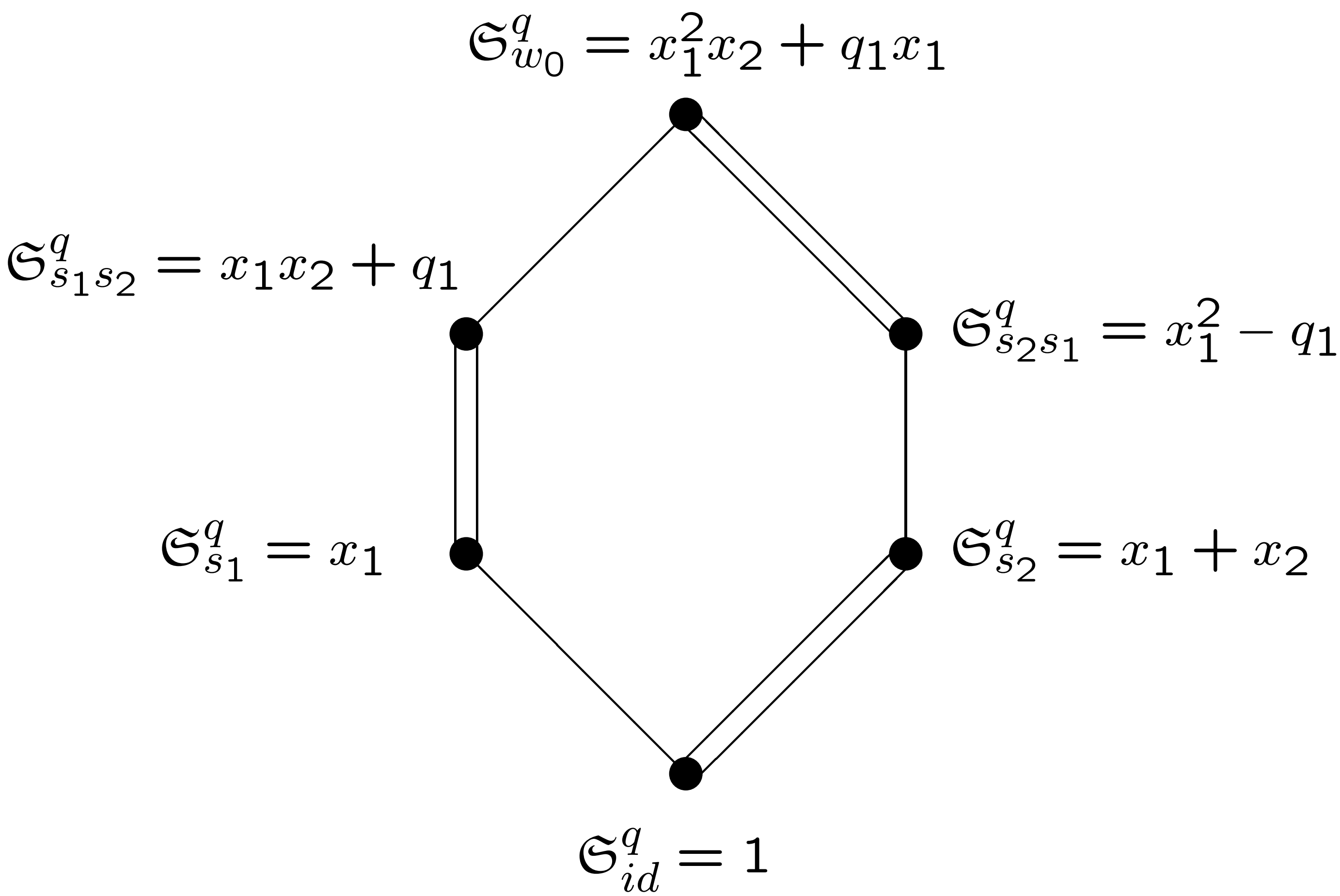
$$\mathrm{QH}^* \simeq \mathbb{Z}[x_1, \ldots, x_n][q_1, \ldots, q_{n-1}]/I_n^q.$$

Example

One can easily calculate the $\mathfrak{S}_w^q$ using the divided differences ∂_i.

$$\mathfrak{S}_{4321} = \mathfrak{S}_{w_0} = e_{123};$$
$$\mathfrak{S}_{3421} = \partial_1 \mathfrak{S}_{w_0} = \partial_1 e_{123} = e_{023};$$
$$\mathfrak{S}_{3412} = \partial_3 e_{023} = (e_2^2)^2 = e_{022} - e_{013}.$$

$$\begin{aligned} &\mathfrak{S}^q_{3412} \\ &= E_{022} - E_{013} \\ &= x_1^2 x_2^2 + 2q_1 x_1 x_2 - q_2 x_1^2 + q_1^2 + q_1 q_2. \end{aligned}$$



Quantum Schubert polynomials for S_3

Axiomatic approach

The following properties of the $\mathfrak{S}_w^q$ follow from their geometric definition:

Axiom 1. Homogeneity: $\mathfrak{S}_w^q$ is a homogeneous polynomial of degree $l(w)$ in $x_1, \ldots, x_n$, $q_1, \ldots, q_{n-1}$, assuming $\deg(x_i) = 1$ and $\deg(q_j) = 2$.

Axiom 2. Classical limit: Specializing $q_1 = \cdots = q_{n-1} = 0$ yields $\mathfrak{S}_w^q = \mathfrak{S}_w$.

Axiom 3. Positivity of GW-invariants:
The c_{uv}^w in

$$\mathfrak{S}_u^q \, \mathfrak{S}_v^q = \sum_w c_{uv}^w \, \mathfrak{S}_w^q$$

are polynomials in the q_i with positive integer coefficients.

Axiom 4. Quantum elementary polynomials:
For a cycle $w = s_{k-i+1} \ldots s_k$, we have

$$\mathfrak{S}_w^q = E_i(x_1, \ldots, x_k).$$

Proved by [Ciocan-Fontanine].

Theorem [FGP] The polynomials $\mathfrak{S}_w^q$ (modulo the ideal I_n^q) are uniquely determined by Axioms 1–4.

Conjecture The polynomials $\mathfrak{S}_w^q$ (mod I_n^q) are uniquely determined by Axioms 1–3.

Checked for S_3 and S_4.

Quantum Monk's formula

Let $t_{ab} = (a, b) = s_a s_{a+1} \dots s_{b-1} \dots s_a$ (transposition).

Theorem [FGP] We have

$$\boxed{\mathfrak{S}^q_{s_r} \mathfrak{S}^q_w = \sum \mathfrak{S}^q_{wt_{ab}} + \sum q_c q_{c+1} \cdots q_{b-1} \mathfrak{S}^q_{wt_{cd}}}$$

where the first sum is over $a \le r < b$ such that $l(wt_{ab}) = l(w) + 1$ and the second sum is over $c \le r < d$ such that $l(wt_{cd}) = l(w) - l(t_{cd})$.

Note that $\mathfrak{S}^q_{s_r} = \mathfrak{S}_{s_r} = x_1 + \cdots + x_r$.

Commuting operators approach

Define the operators on $K[x_1, x_2, \ldots]$

$$X_k = x_k - \sum_{i<k} q_{ij}\partial_{(ij)} + \sum_{j>k} q_{kj}\partial_{(kj)}$$

where $\partial_{(ij)} = \partial_i\partial_{i+1}\ldots\partial_{j-1}\ldots\partial_{i+1}\partial_i$
and $q_{ij} = q_i q_{i+1}\cdots q_{j-1}$.

Theorem [FGP]

- The operators X_k commute pairwise and $K[X_1, X_2, \ldots]$ is a free abelian group.

- For any $g \in K[x_1, x_2, \ldots]$ there is a unique polynomial $G \in K[X_1, X_2, \ldots]$ such that $G : 1 \mapsto g$.

- The map $g \mapsto G$ is the quantization map ψ. In particular, $e_I \mapsto E_I$ and $\mathfrak{S}_w \mapsto \mathfrak{S}_w^q$.

- X_i induces the operator of quantum multiplication by x_i in $\mathbb{Z}[x_i, q_j]/I_n \simeq \mathsf{H}^* \otimes \mathbb{Z}[q_j]$.

Examples:

$$
\begin{aligned}
X_i(1) &= x_i, \\
X_1X_1(1) &= x_1^2 + q_1, \\
X_iX_i(1) &= x_i^2 - q_{i-1} + q_i, \quad i > 1 \\
X_iX_{i+1}(1) &= X_{i+1}X_i(1) = x_ix_{i+1} - q_i, \\
X_1X_1X_1(1) &= x_1^3 + 2q_1x_1 + q_1x_2.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\psi:\ & x_i && \longmapsto && x_i, \\
\psi:\ & x_1^2 && \longmapsto && x_1^2 - q_1, \\
\psi:\ & x_i^2 && \longmapsto && x_i^2 + q_{i-1} - q_i, \quad i > 1 \\
\psi:\ & x_ix_{i+1} && \longmapsto && x_ix_{i+1} + q_i, \\
\psi:\ & x_1^3 && \longmapsto && x_1^3 - 2q_1x_1 - q_1x_2.
\end{aligned}
$$

Three definitions of $\mathfrak{S}_w^q$:

1. $\mathfrak{S}_w^q$ represents σ_w in QH*.

2. Quantization map $\psi : e_I \mapsto E_I$.

3. $\psi : g(x_1, x_2, \dots) \mapsto G(X_1, X_2, \dots)$.