Permutohedra, Associahedra, and Beyond

or

Three Formulas for Volumes of Permutohedra

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on the occasion of Richard P. Stanley's Birthday

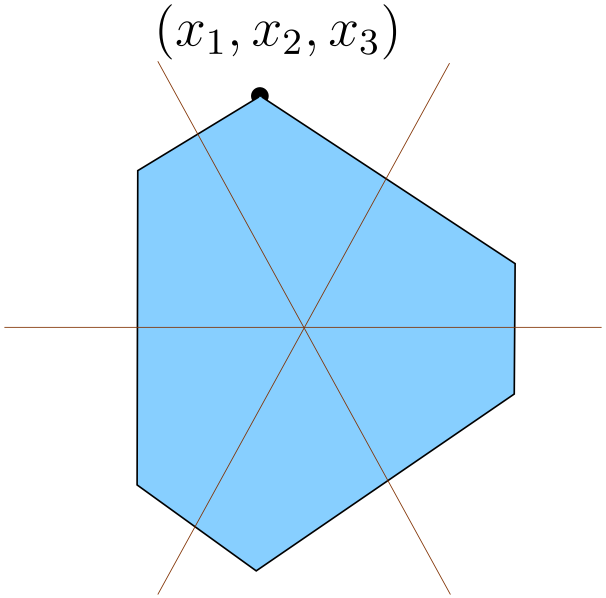
Permutohedron

$$P_n(x_1, \ldots, x_{n+1}) := \mathrm{ConvexHull}((x_{w(1)}, \ldots, x_{w(n+1)}) \mid w \in S_{n+1})$$

This is a convex n-dimensional polytope in $H \subset \mathbb{R}^{n+1}$.

Example: $n = 2$ (type A_2)

$P_2(x_1, x_2, x_3) =$



More generaly, for a Weyl group W, $P_W(x) := \mathrm{ConvexHull}(w(x) \mid w \in W)$.

Question: What is the volume $V_n := \operatorname{Vol} P_n$?

Volume form is normalized so that the volume of a parallelepiped formed by generators of the lattice $\mathbb{Z}^{n+1} \cap H$ is 1.

Question: What is the number of lattice points $N_n := P_n \cap \mathbb{Z}^{n+1}$?

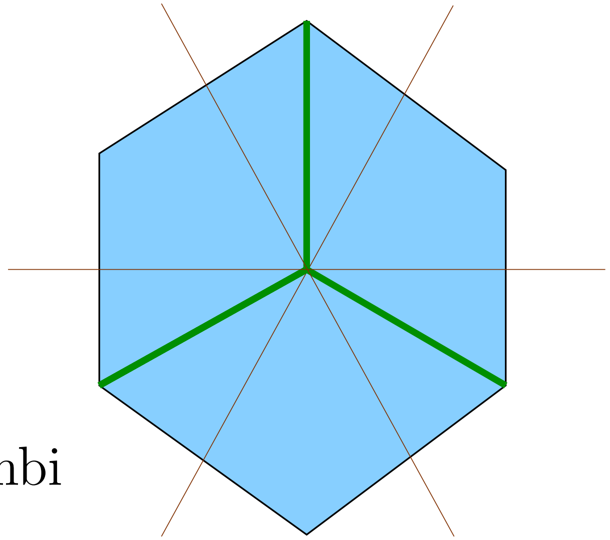
We will see that V_n and N_n are polynomials in $x_1, \ldots, x_{n+1}$ of degree n. The polynomial V_n is the top homogeneous part of N_n. The Ehrhart polynomial of P_n is $E(t) = N_n(tx_1, \ldots, tx_n)$, and V_n is its top coefficient.

We will give 3 totally different formulas for these polynomials.

Special Case:

$$P_n(n+1, n, \ldots, 1) = \text{ConvexHull}((w(1), ..., w(n+1)) \mid w \in S_{n+1})$$

is the most symmetric permutohedron.



regular hexagon

subdivided into 3 rhombi

It is a zonotope = Minkowsky sum of line intervals.

Well-known facts:

- $V_n(n+1, \ldots, 1) = (n+1)^{n-1}$ is the number of trees on $n+1$ labelled vertices. $P_n(n+1, \ldots, 1)$ can be subdivided into parallelepipeds of unit volume associated with trees. This works for any zonotope.
- $N_n(n+1, \ldots, 1)$ is the number of forests on $n+1$ labelled vertices.

First Formula

Fix any distinct numbers $\lambda_1, \ldots, \lambda_{n+1}$ such that $\lambda_1 + \cdots + \lambda_{n+1} = 0$.

$$V_n(x_1, \ldots, x_{n+1}) = \frac{1}{n!} \sum_{w \in S_{n+1}} \frac{(\lambda_{w(1)} x_1 + \cdots + \lambda_{w(n+1)} x_{n+1})^n}{(\lambda_{w(1)} - \lambda_{w(2)})(\lambda_{w(2)} - \lambda_{w(3)}) \cdots (\lambda_{w(n)} - \lambda_{w(n+1)})}$$

Notice that the symmetrization in RHS does not depends on λ_i's.

Idea of Proof Use Khovansky-Puchlikov's method:

- Instead of just counting the number of lattice points in P, define $[P] =$ sum of formal exponents e^a over lattice points $a \in P \cap \mathbb{Z}^n$.
- Now we can work with unbounded polytopes. For example, for a simplicial cone C, the sum $[C]$ is given by a simple rational expression.
- Any polytope P can be explicitly presented as an alternating sum of simplicial cones: $[P] = [C_1] \pm [C_2] \pm \cdots$.

Applying this procedure to the permutohedron, we obtain ...

Let $\alpha_1, \ldots, \alpha_n$ be a system of simple roots for Weyl group W, and let L be the root lattice.

Theorem: *For a dominant weight* μ*,*

$$[P_W(\mu)] := \sum_{a \in P_W(\mu) \cap (L+\mu)} e^a = \sum_{w \in W} \frac{e^{w(\mu)}}{(1 - e^{-w(\alpha_1)}) \cdots (1 - e^{-w(\alpha_n)})}$$

Compare this with Weyl's character formula!

Note: LHS is obtained from the character $ch\ V_\mu$ of the irrep V_μ by replacing all nonzero coefficients with 1. In type A, $ch\ V_\mu =$ Schur polynomial s_μ.

From this expression, one can deduce the First Formula and also its generalizations to other Weyl groups.

Second Formula

Let us use the coordinates $y_1, \ldots, y_{n+1}$ related $x_1, \ldots, x_{n+1}$ by

$$\begin{cases} y_1 = -x_1 \\ y_2 = -x_2 + x_1 \\ y_3 = -x_3 + 2x_2 - x_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_{n+1} = -\binom{n}{0} x_n + \binom{n}{1} x_{n-1} - \cdots \pm \binom{n}{n} x_1 \end{cases}$$

and write $V_n = \mathrm{Vol}\ P_n(x_1, \ldots, x_{n+1})$ as a polynomial in $y_1, \ldots, y_{n+1}$.

Examples:

$$V_1 = \mathrm{Vol}\left([(x_1, x_2), (x_2, x_1)]\right) = x_1 - x_2 = y_2$$

$$V_2 = \cdots = 3\, y_2^2 + 3\, y_2\, y_3 + \tfrac{1}{2}\, y_3^2$$

Theorem:

$$\boxed{V_n(x_1,\ldots,x_{n+1}) = \frac{1}{n!}\sum_{(S_1,\ldots,S_n)} y_{|S_1|}\cdots y_{|S_n|},}$$

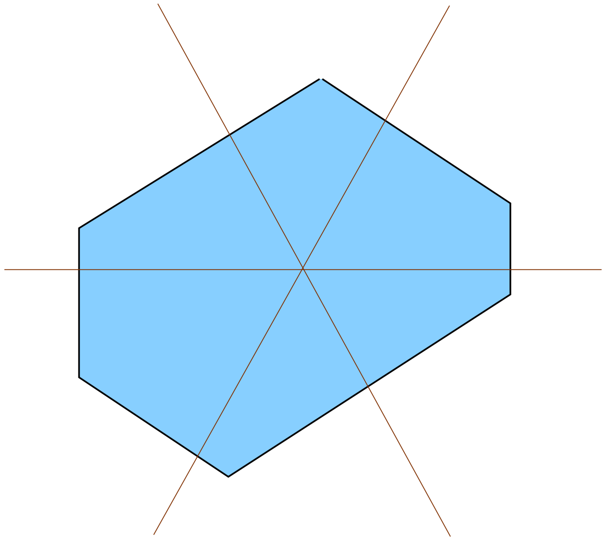
where the sum is over ordered collections of subsets $S_1,\ldots,S_n \subset [n+1]$ such that either of the following equivalent conditions is satisfied:

- For any distinct $i_1,\ldots,i_k$, we have $|S_{i_1}\cup\cdots\cup S_{i_k}| \geq k+1$ (cf. Hall's Marriage Theorem)
- For any $j \in [n+1]$, there is a system of distinct representatives in $S_1,\ldots,S_n$ that avoids j.

Thus $n!\,V_n$ is a polynomial in $y_2,\ldots,y_{n+1}$ with positive integer coefficients.

This formula can be extended to generalized permutohedra

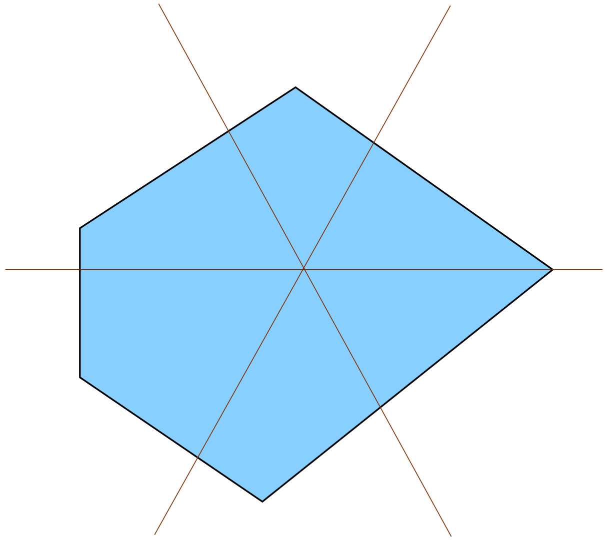
a generalized permutohedron



Generalized permutohedra are obtained from usual permutohedra by moving faces while preserving all angles.

this is also

a generalized permutohedron



Generalized Permutohedra

Coordinate simplices in $\mathbb{R}^{n+1}$: $\Delta_I = \mathrm{ConvexHull}(e_i \mid i \in I)$, for $I \subseteq [n+1]$.
Let $\mathbf{Y} = \{Y_I\}$ be the collection of variables $Y_I \geq 0$ associated with all subsets $I \subset [n+1]$. Define

$$P_n(\mathbf{Y}) := \sum_{I \subset [n+1]} Y_I \cdot \Delta_I \qquad \text{(Minkowsky sum)}$$

Its combinatorial type depends only on the set of I's for which $Y_I \neq 0$.

Examples:

- If $Y_I = y_{|I|}$, then $P_n(\mathbf{Y})$ is a usual permutohedron.
- If $Y_I \neq 0$ iff I is a consecutive interval, then $P_n(\mathbf{Y})$ is an associahedron.
- If $Y_I \neq 0$ iff I is a cyclic interval, then $P_n(\mathbf{Y})$ is a cyclohedron.
- If $Y_I \neq 0$ iff I is a connected set in Dynkin diagram, then $P_n(\mathbf{Y})$ is a generalized associahedron related to DeConcini-Procesi's work.
  (Do not confuse with Fomin-Zelevinsky's generalized associahedra!)
- If $Y_I \neq 0$ iff I is an initial interval $\{1, \ldots, i\}$, then $P_n(\mathbf{Y})$ is the Stanley-Pitman polytope.

Theorem: The volume of the generalized permutohedron is given by

$$\boxed{\mathrm{Vol}\, P_n(\mathbf{Y}) = \frac{1}{n!} \sum_{(S_1,\dots,S_n)} Y_{S_1} \cdots Y_{S_n},}$$

where $S_1, \dots, S_n$ satisfy the same condition.

Theorem: The # of lattice points in the generalized permutohedron is

$$\boxed{P_n(\mathbf{Y}) \cap \mathbb{Z}^{n+1} = \frac{1}{n!} \sum_{(S_1,\dots,S_n)} \{Y_{S_1} \cdots Y_{S_n}\},}$$

$$\left\{\prod_I Y_I^{a_I}\right\} := (Y_{[n+1]}+1)^{\{a_{[n+1]}\}} \prod_{I \neq [n+1]} Y_I^{\{a_I\}}, \text{ where } Y^{\{a\}} = Y(Y+1)\cdots(Y+a-1).$$

This extends a formula from [Stanley-Pitman] for the volume of their polytope. In this case, the above summation is over parking functions.

We also have a combinatorial description of face structure of generalized permutohedra in terms of nested collections of subsets in $[n+1]$. This is related to DeConcini-Procesi's wonderful arrangements.

Not enough time for this now.

The most interesting part of the talk is ...

Third Formula

Let use the coordinates $z_1, \ldots, z_n$ related to $x_1, \ldots, x_{n+1}$ by

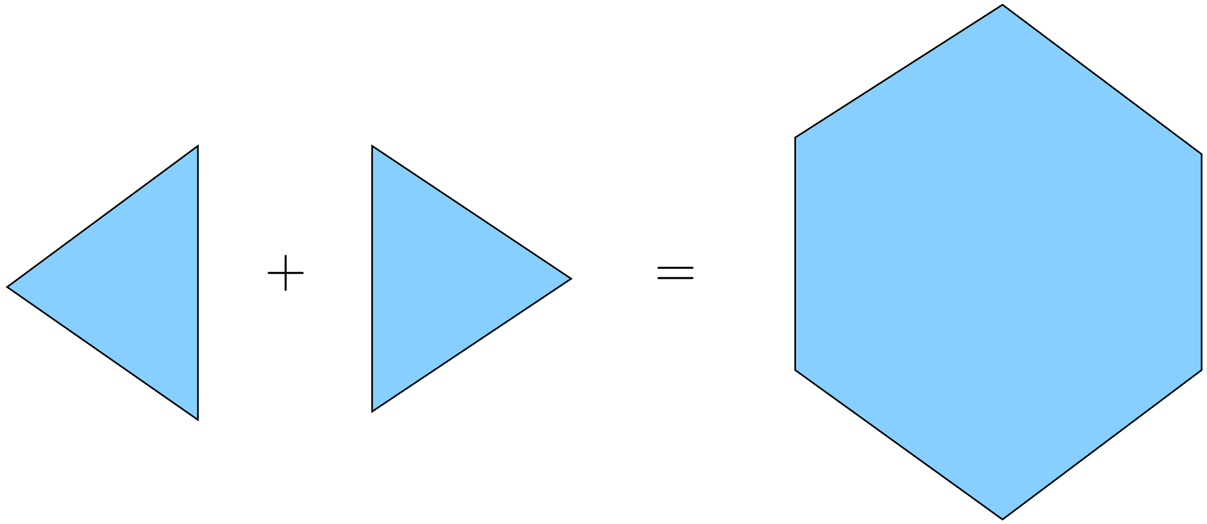
$$z_1 = x_1 - x_2,\ z_2 = x_2 - x_3,\ \cdots,\ z_n = x_n - x_{n+1}$$

These coordinates are canonically defined for an arbitrary Weyl group.

Then the permutohedron P_n is the Minkowsky sum

$$P_n = z_1\,\Delta_{1n} + z_2\,\Delta_{2n} + \cdots + z_n\,\Delta_{nn}$$

of hypersimplices $\Delta_{kn} = P_n(1, \ldots, 1, 0, \ldots, 0)$ (with k 1's).



This implies

$$\mathrm{Vol}\ P_n = \sum_{c_1,\dots,c_n} A_{c_1,\dots,c_n} \frac{z_1^{c_1}}{c_1!}\cdots\frac{z_n^{c_n}}{c_n!},$$

where the sum is over $c_1,\dots,c_n \geq 0$, $c_1+\cdots+c_n = n$, and

$$A_{c_1,\dots,c_n} = \mathrm{MixedVolume}(\Delta_{1n}^{c_1},\dots,\Delta_{nn}^{c_n}) \in \mathbb{Z}_{>0}$$

In particular, $n!\,V_n$ is a positive integer polynomial in $z_1,\dots,z_n$.

Let us call the integers $A_{c_1,\dots,c_n}$ the Mixed Eulerian numbers.

Examples:

$$V_1 = 1\, z_1$$

$$V_2 = 1\,\frac{z_1^2}{2} + 2\,z_1z_2 + 1\,\frac{z_2^2}{2}$$

$$V_3 = 1\,\frac{z_1^3}{3!} + 2\,\frac{z_1^2}{2}z_2 + 4\,z_1\frac{z_2^2}{2} + 4\,\frac{z_2^3}{3!} + 3\,\frac{z_1^2}{2}z_3 + 6\,z_1z_2z_3 + \\ +4\,\frac{z_2^2}{2}z_3 + 3\,z_1\frac{z_3^2}{2} + 2\,z_2\frac{z_3^2}{2} + 1\,\frac{z_3^3}{3!}$$

(The mixed Eulerian numbers are marked in red.)

Properties of Mixed Eulerian numbers:

- $A_{c_1,\dots,c_n}$ are positive integers defined for $c_1,\dots,c_n \geq 0$, $c_1+\cdots+c_n = n$.
- $\sum \frac{1}{c_1!\cdots c_n!} A_{c_1,\dots,c_n} = (n+1)^{n-1}$.
- $A_{0,\dots,0,n,0,\dots,0}$ (n is in k-th position) is the usual Eulerian number A_{kn} $=$ # permutations in S_n with k descents $= n! \operatorname{Vol} \Delta_{kn}$.
- $A_{1,\dots,1} = n!$
- $A_{k,0,\dots,0,n-k} = \binom{n}{k}$
- $A_{c_1,\dots,c_n} = 1^{c_1} 2^{c_2} \cdots n^{c_n}$ if $c_1+\cdots+c_i \geq i$, for $i = 1,\dots,n$.
  There are exactly $C_n = \frac{1}{n+1}\binom{2n}{n}$ such sequences $(c_1,\dots,c_n)$.

When I showed these numbers to Richard Stanley, he conjectured that

- $\sum A_{c_1,\dots,c_n} = n!\, C_n$.

Moreover, he conjectured that ...

One can subdivide all sequences $(c_1, \ldots, c_n)$ into C_n classes such that the sum of mixed Eulerian numbers for each class is $n!$. For example, $A_{1,\ldots,1} = n!$ and $A_{n,0,\ldots,0} + A_{0,n,0,\ldots,0} + A_{0,0,n,\ldots,0} + \cdots + A_{0,\ldots,0,n} = n!$, because the sum of Eulerian numbers $\sum_k A_{kn}$ is $n!$.

Let us write $(c_1, \ldots, c_n) \sim (c'_1, \ldots, c'_n)$ iff $(c_1, \ldots, c_n, 0)$ is a cyclic shift of $(c'_1, \ldots, c'_n, 0)$. Stanley conjectured that, for fixed $(c_1, \ldots, c_n)$, we have

$$\sum_{(c'_1,\ldots,c'_n)\sim(c_1,\ldots,c_n)} A_{c'_1,\ldots,c'_n} = n!$$

Exercise: *Check that there are exactly C_n equivalence classes of sequences.*

Every equivalence class contains exactly one sequence $(c_1, \ldots, c_n)$ such that $c_1 + \cdots + c_i \geq i$, for $i = 1, \ldots, n$. (For this sequence, $A_{c_1,\ldots,c_n} = 1^{c_1} \cdots n^{c_n}$.)

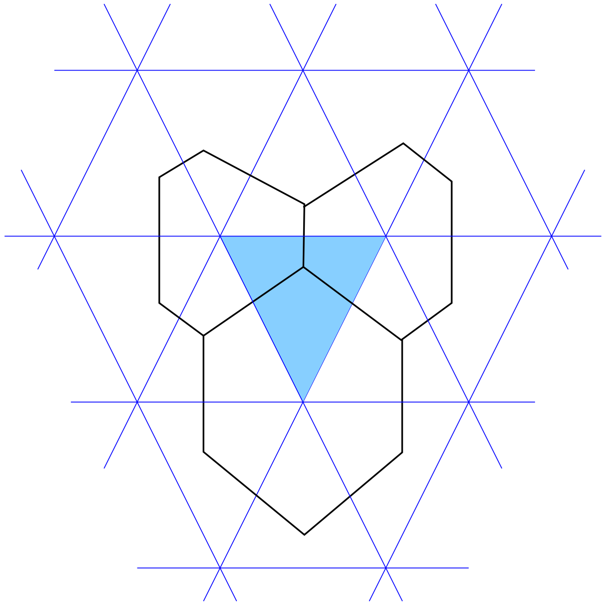
These conjectures follow from ...

Theorem: *Let* $U_n(z_1, \ldots, z_{n+1}) = \mathrm{Vol}\, P_n$. *(It does not depend on* z_{n+1}*.)*

$$U_n(z_1, \ldots, z_{n+1}) + U_n(z_{n+1}, z_1, \ldots, z_n) + \cdots + U_n(z_2, \ldots, z_{n+1}, z_1) = \\ = (z_1 + \cdots + z_{n+1})^n$$

This theorem has a simple geometric proof. It extends to any Weyl group. Cyclic shifts come from symmetries of type A extended Dynkin diagram.

Idea of Proof:



The area of blue triangle is $\frac{1}{6}$ sum of the areas of three hexagons.

Corollary: Fix $z_1, \ldots, z_{n+1}$, $\lambda_1, \ldots, \lambda_{n+1}$ such that $\lambda_1 + \cdots + \lambda_{n+1} = 0$. Symmetrizing the expression

$$\frac{1}{n!} \frac{(\lambda_1 z_1 + (\lambda_1 + \lambda_2) z_2 + \cdots (\lambda_1 + \cdots + \lambda_{n+1}) z_{n+1})^n}{(\lambda_1 - \lambda_2) \cdots (\lambda_n - \lambda_{n+1})}$$

with respect to $(n+1)!$ permutations of $\lambda_1, \ldots, \lambda_{n+1}$ and $(n+1)$ cyclic permutations of $z_1, \ldots z_{n+1}$, we obtain

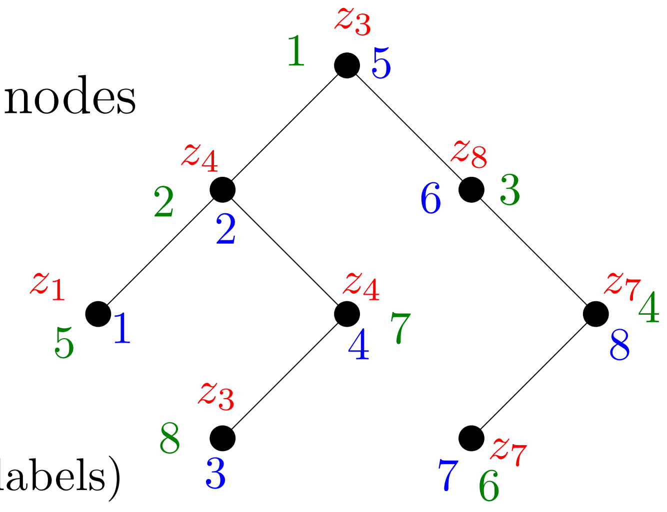
$$(z_1 + \cdots + z_{n+1})^n.$$

Problem: *Find a direct proof.*

Combinatorial interpretation for $A_{c_1,\dots,c_n}$

a plane binary tree on n nodes

$T =$



$z^T = z_3 z_4 z_8 z_7 z_1 z_7 z_4 z_3$

(the order is given by green labels)

- The nodes are labelled by $1, \dots, n$ such that, for a node labelled l, labels of all in the left (right) branch are less (greater) than l. The labels of all descendants of a node form a consecutive interval $I = [a, b]$.
- We have an increasing labelling of the nodes by $1, \dots, n$.
- Each node is labeled by z_i such that $i \in I$; $z^T :=$ product of all z_i's.
- The weight of a node labelled by l and z_i with interval $[a, b]$ is $\frac{i-a+1}{l-a+1}$ if $i \leq l$, and $\frac{b-i+1}{b-l+1}$ if $i \geq l$. The weight $wt(T)$ of tree is the product of weights of its nodes.

Theorem: *The volume of the permutohedron is*

$$\boxed{V_n = \sum_T wt(T) \cdot z^T}$$

where the sum is over plane binary trees with blue, red, and green labels.

Combinatorial interpretation for the mixed Eulerian numbers:

Theorem: *Let* $z_{i_1} \cdots z_{i_n} = z_1^{c_1} \cdots z_n^{c_n}$. *Then*

$$A_{c_1,\dots,c_n} = \sum_T n!\, wt(T)$$

over same kind of trees T *such that* $z^T = z_{i_1} \cdots z_{i_n}$ *(in this order).*

Note that all terms $n!\, wt(T)$ in this formula are positive integer.

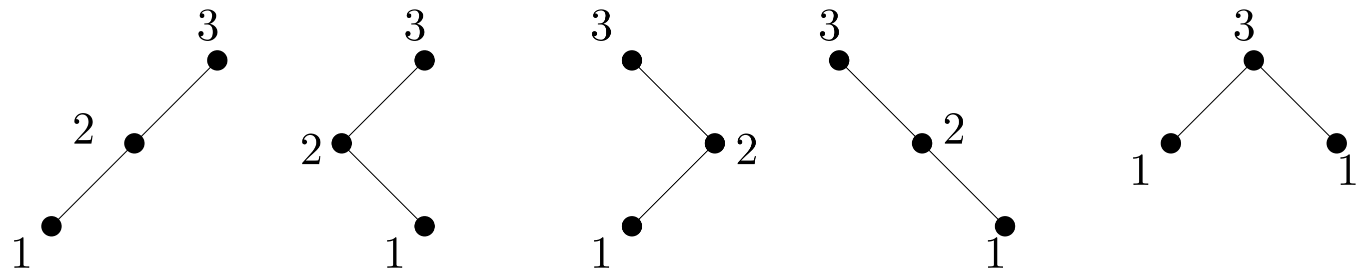
Comparing different formulas for V_n, we obtain a lot of interesting combinatorial identities. For example ...

Corollary:

$$(n+1)^{n-1} = \sum_T \frac{n!}{2^n} \prod_{v \in T} \left(1 + \frac{1}{h(v)}\right),$$

where is sum is over unlabeled plane binary trees T on n nodes, and $h(v)$ denotes the "hook-length" of a node v in T, i.e., the number of descendants of v (including v).

Example: $n = 3$



hook-lengths of binary trees

The identity says that

$$(3+1)^2 = 3 + 3 + 3 + 3 + 4.$$

Problem: *Prove this identity combinatorially.*